Tutorial on Attached-Mean Axes and Their Use in the Calculation of Deformable Static and Damped-Undamped Vibration Modes of a Free-Free Structure

Paolo Mantegazza[1]

(Received: Dec 6, 2010. Revised: Feb 7, 2011. Accepted: Feb 18, 2011)

1. Introduction

The vibration modes of a free structure can be divided into rigid and deformable modes. Rigid modes can be related to the whole structure or just to parts of it, e.g. an aircraft and its control surfaces. Rigid modes can be associated to real motions of the type just mentioned, or to local/global indeterminacies in the static structural response. The latter imply hidden mechanisms to which we can associate spurious rigid kinematic modes, or simply kinematic modes in the following. Rigid and kinematic modes are clearly the same thing but we use the two terms in such a way that rigid modes indicates expected rigid motions, known or definable a priory as such, while kinematic modes will refer to unexpected rigid motions due to a wrong design, hopefully a rare case, or, more often the case, to modeling errors. In such a view the calculation of kinematic modes is an important step in validating a model, as their appearance addresses design/modeling errors to be compulsorily fixed before undertaking any further serious analysis [4, 23, 24].

The numerical determination of vibration modes is based, regardless of how it is obtained, on a discretized equation of the type:

$$\boldsymbol{K}\,\boldsymbol{u} = \omega^2 \boldsymbol{M}\,\boldsymbol{u}. \tag{1}$$

with the stiffness matrix $\boldsymbol{K}$ being semidefinite positive as many times as there are rigid-kinematic modes.
For a non singular $\boldsymbol{K}$ it is well known that a power filtering iteration of the type:

$$\boldsymbol{U}^{(k+1)} = \boldsymbol{K}^{-1}\boldsymbol{M}\,\boldsymbol{U}^{(k)} \tag{2}$$

applied to a subspace of vectors $\boldsymbol{U}$ enriches it toward the components associated to the lowest frequency modes [3, 25]. As such it is the core of many of the methods used for the calculation of a subset of low frequency vibration modes of large problems, e.g. block-subspace and Lanczos methods [22]. It is noted also that, to preserve sparsity as much as possible and to improve numerical conditioning, the power filtering is always implemented without any inversion. Instead, the solution of

$$\boldsymbol{K}\,\boldsymbol{U}^{(k+1)} = \boldsymbol{M}\,\boldsymbol{U}^{(k)} \tag{3}$$

is used, through a single factorization of $\boldsymbol{K}$ and repeated forward-backward substitutions.

A somewhat standard general technique [5, 6] to preserve sparsity and obviate to any singularity of Eq. (3), so allowing the calculation of rigid-kinematic modes also, is to change the origin of ω^2, writing

$$(\boldsymbol{K} + \alpha\boldsymbol{M})\boldsymbol{u} = (\omega^2 + \alpha)\boldsymbol{M}\,\boldsymbol{u} \tag{4}$$

$$(\boldsymbol{K} + \alpha\boldsymbol{M})\boldsymbol{U}^{(k+1)} = \boldsymbol{M}\;\boldsymbol{U}^{(k)} \tag{5}$$

[1] Dipartimento Ingegneria Aerospaziale, Politecnico di Milano

doi:10.3293/asdj.2011.12

with $\alpha > 0$ to avoid the risk of making the modified stiffness matrix, $(\boldsymbol{K} + \alpha\boldsymbol{M})$, negative definite. $\boldsymbol{M}$ should be positive definite always but, e.g. finite elements with rotational degrees of freedom and lumped linear masses only, it could even be semidefinite. Such a case implies that there are purely static degrees of freedom, which can be eliminated in an algebraic manner, so becoming dependent on true dynamic degrees of freedom only, for which $\boldsymbol{M}$ is positive definite for sure. Eq. (4) can nonetheless be used as it is because the algebraic only degrees of freedom will produce infinite frequencies that will not disturb the related shifted power iteration, since it aims at a subspace of a relatively limited number of low frequencies only.

Numerical reasons suggest choosing α as a suitable fraction of the best estimate available for the lowest deformable ω^2. By not distinguishing rigid-kinematic from deformable modes such an approach can produce imprecise rigid body modes and affect the precision of the, possibly closed to zero, lowest deformable frequencies [14]. Nonetheless, the long standing experience gathered so far has shown that the above shift approach is reliable enough for most practical applications. It is here thought that it remains one of the best ways to validate a finite element model in relation to the search of hidden kinematic modes, while simultaneously confirming the expected existence of rigid body modes. Once found, verified and accepted as true rigid modes any of the latter can often be subsequently reassigned, very precisely, by inspection.

An alternative way to determine rigid-kinematic modes is to search for the null space of $\boldsymbol{K}$. Within such a process if only whole structure rigid body modes existed they will appear at the factorization of the very last equations, i.e. those corresponding to the number of true overall rigid modes. So any singularity stepping into the factorization before them will be an indication of the presence of an internal rigid-kinematic mode. In such a view local constraints can be applied each time a singularity appears, continuing the factorizations till all rigid-kinematic modes are evidenced. Such a process can be carried out with partial diagonal pivoting to preserve symmetry and sparsity, as far as possible with respect to numerical stability [7, 25, 26].

From the singularities reported during the factorization one can then separate true rigid body from kinematic modes and proceed to the same design-modeling fixes as it should have been done with a verification using the shifted eigen calculation previously addressed. It should be remarked that the search of the null space of $\boldsymbol{K}$ can be worked out in a sound numerical way also by using a Singular Value Decomposition (SVD) [13], but such an approach is not viable for the kind of large and sparse Finite Element (FE) problems quite common nowadays.

A final option is to apply a shift to the stiffness matrix only and then proceed to a power iteration with it. Such a technique is equivalent to computing the null frequency eigenmodes with an appropriate fictitious lumped scalar mass matrix, chosen so to alleviate possible problems associated to null frequencies obtained with the real mass.

On the base of what said it is hereafter assumed that the calculation of deformable vibration modes can then proceed by assuming the availability of a fully validated elasto-dynamic model and numerically well known true rigid body modes.

Within such a framework the aim becomes that of calculating deformable modes only, with a sound formulation and without the need of using any frequency shift. Such an approach is not used often though, likely because the simple shifted scheme of Eq. (4) mostly gives satisfactory results. It has nonetheless the potential for improving vibration modes analyses in the case of closely clustered deformable lowest frequency modes not too far away from zero, a not so rare case for very large and complex structures, e.g. modern jumbo aerospace vehicles.

In this tutorial we will present an approach based on the use of mean axes that leads to the separation of the eigenproblem associated to rigid and deformable modes in a simple and neat way. Moreover, it unifies deformable vibration and static displacements and modes calculations for free structures, e.g. such as for the inertia relief and static residualizations of low order modal dynamic responses (modes acceleration). To such an aim, as suggested by the title, we will take it at large by introducing a broader view of attached-mean axes approximations, hoping it might be of interest for a more general modeling usage also.

2. Framing the calculation of deformable vibration modes through attached-mean axes

Attached-mean axes refers to a scheme for approximating the motion of a free deformable structure by separating its overall motion into a rigid and a deformable part. As mentioned in [17], where the classification attached-mean-principal axes is given, the idea of mean axes dates back to [15]. Following it the concept has been reprized in many forms, [9] should be consulted for one of the many interpretation within the more general concept of a Tisserand frame. Within such a framework it will be seen that attached and mean axes differs only in the representation of generalized external forces and the mass matrix, no difference being produced for the elastic internal forces, i.e. the stiffness and structural damping matrices.

For general and arbitrarily large motions mean axes can produce only a very limited decoupling of the inertia forces pertaining to the whole reference frame motion from those associated to deformable motions only. In fact their adoption is often criticized on the base of their uselessness for general dynamic simulations [16].

While being not out of place completely the associated criticism is undue when applied to small motions of a free structure around a steady straight trajectory, where mean axes produce a significant simplification by wholly separating inertia forces related to the reference frame from deformable motions, so that the equations of motion of the reference frame remain the same as those of the related rigid body. As such they will remain very useful as long as the mentioned linear equations of motion of freely moving aerospace vehicles will continue to be of interest to engineers.

So we now reprise the development of discretized attached-mean axes models for a linear elastic continuum by using the somewhat standard combination of the linear Principle of Virtual Work (PVW) [18]

$$\begin{aligned}\int_V \delta\boldsymbol{\varepsilon}^T\ \boldsymbol{\sigma} dV = \int_V \delta\boldsymbol{s}^T \boldsymbol{f}_v dV + \int_S \delta\boldsymbol{s}^T \boldsymbol{f}_s dS +\\ \int_V \delta\boldsymbol{\phi}^T \boldsymbol{c}_v dV + \int_S \delta\boldsymbol{\phi}^T \boldsymbol{c}_s dS - \int_V \delta\boldsymbol{s}^T \boldsymbol{m}\ \boldsymbol{a}\ dV \end{aligned} \tag{6}$$

$$\boldsymbol{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}\}^T \tag{7}$$

$$\boldsymbol{\varepsilon} = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{13}, 2\epsilon_{23}\}^T, \tag{8}$$

and a complete base Ritz Approximation (RA) of the displacement $\boldsymbol{s}(\boldsymbol{x}, t)$:

$$\begin{aligned}\boldsymbol{s} = \begin{Bmatrix} \boldsymbol{d}(\boldsymbol{x}, t) \\ \boldsymbol{\phi}(\boldsymbol{x}, t) \end{Bmatrix} = \begin{bmatrix} \boldsymbol{I} & -(\boldsymbol{x} - \boldsymbol{x}_o)\times & \boldsymbol{v}_\delta \times (\boldsymbol{x} - \boldsymbol{x}_p) & \boldsymbol{N}_d(\boldsymbol{x}) \\ 0 & \boldsymbol{I} & \boldsymbol{v}_\delta & \boldsymbol{N}_\phi(\boldsymbol{x}) \end{bmatrix} \begin{Bmatrix} \boldsymbol{o}(t) \\ \boldsymbol{\theta}(t) \\ \boldsymbol{\delta}(t) \\ \boldsymbol{q}_a(t) \end{Bmatrix} =\\ \begin{bmatrix} \boldsymbol{R}(\boldsymbol{x}) & \boldsymbol{N}_a(\boldsymbol{x}) \end{bmatrix} \begin{Bmatrix} \boldsymbol{r}_a(t) \\ \boldsymbol{q}_a(t) \end{Bmatrix}\end{aligned} \tag{9}$$

with: $\boldsymbol{\sigma}$ the Cauchy stresses, $\boldsymbol{\varepsilon}$ the small deformations,

$$\boldsymbol{r}_a = \begin{Bmatrix} \boldsymbol{o} \\ \theta \\ \boldsymbol{\delta} \end{Bmatrix}, \ \boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} & -(\boldsymbol{x} - \boldsymbol{x}_o)\times & \boldsymbol{v}_\delta \times (\boldsymbol{x} - \boldsymbol{x}_p) \\ 0 & \boldsymbol{I} & \boldsymbol{v}_\delta \end{bmatrix}, \ \boldsymbol{N}_a = \begin{bmatrix} \boldsymbol{N}_d(\boldsymbol{x}) \\ \boldsymbol{N}_\phi(\boldsymbol{x}) \end{bmatrix} \quad (10)$$

where we have assumed it convenient considering control surfaces rotations, $\boldsymbol{\delta}$, through their pivot point, $\boldsymbol{x}_p$, as rigid modes. In the above equations we have chosen to specify the rigid motion of the whole structure as three translations of a point $\boldsymbol{x}_o$ and three rotations about axes through that point. Clearly any linear combination of base functions is a base, so that any other rigid motion representation could serve our scope. It should be noticed that the deformable displacement field can represent both linear, $\boldsymbol{N}_d(\boldsymbol{x})$, and small rotations, $\boldsymbol{N}_\phi(\boldsymbol{x})$. That is done to take into account the external volume and surface forces, $\boldsymbol{f}_v$ and $\boldsymbol{f}_s$, and couples, $\boldsymbol{c}_v$ and $\boldsymbol{c}_s$, as well as a generalized mass distribution represented by a (6x6) local mass matrix $\boldsymbol{m}$, i.e. having mass, static moments and moments of inertia. On the other hand we will not extend a rotational formulation to the internal stress-deformation field, which will remain that of a standard non polar continuum. For a notational simplification we have included distributed external loads and mass matrix distributions only, possible concentrated distributions being accounted for by appropriately placed delta functions (Dirac δ).

It is remarked that the rigid-deformable separation needs not imply describing the deformable motions in term of coordinates relative to such a frame. They can be taken in absolute coordinates also, what counts is just making the separation explicit. In fact they are the same in the case of overall small motions, differences being second order [1]. Even if they are based on the same discretization scheme and tools, FE models differ from their attached-mean axes counterparts because all of their motion discretization is taken in term of absolute nodal displacements and rotations, while attached-mean axes schemes are based on describing deformable motions with respect to an attached-mean frame. It will be seen that FE can be easily translated to attached-mean axes, albeit at the expense of loosing some, in the case of attached axes, or all, in the case of mean axes, of the sparsity of their equations. It is for such a reason that attached-mean axes are mostly used with globally instead than with locally based approximating functions. So they are mostly viable for reduced model, often derived from FE schemes, e.g. when vibration and/or global selected static deformable modes are used.

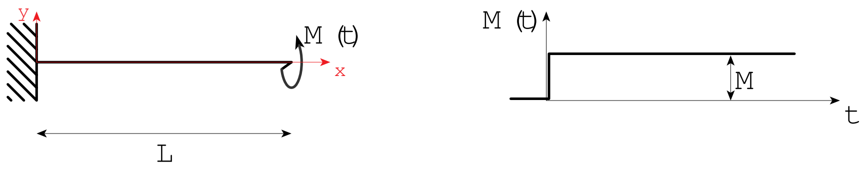
Then, beginning with attached axes, we impose that the deformable shape functions $\boldsymbol{N}_d(\boldsymbol{x})$ must not contain any rigid body motion. Such a convenient constraint is not strictly needed, as long as they are independent functions of a complete set with non null strains. Nonetheless we prefer that they represent a complete set of statically determinate displacement functions, from which it follows their qualification as "attached". It should be noticed that the fact they are statically determinate implies that the motion, relative to the moving frame, of a few appropriate points of the space containing the structure are null. Nonetheless such points need not to pertain to the material part of the structure but just to the, attached, domain of definition of $\boldsymbol{N}_d(\boldsymbol{x})$ and $\boldsymbol{N}_\phi(\boldsymbol{x})$.

Whatever to choice of the rigid body modes they need not to be associated to the rigid modes corresponding to the constraints they are attached to. So, in our case, the structure must not be compulsorily built in at the axes origin but can be attached to any set of determinate constraints. $\boldsymbol{N}_d(\boldsymbol{x})$ and $\boldsymbol{N}_\phi(\boldsymbol{x})$ must be complete sets, in the sense that they will however converge, independently from the applied loads, to the true solution as their number is increased. On the other hand they need not satisfy natural boundary conditions, as they will

[1] Assuming $\boldsymbol{N}_d\ \boldsymbol{q}_a$ is a small deformable displacement relative to the reference frame, its absolute counterpart $\boldsymbol{s}_{abs}$ for a small rotation $\boldsymbol{\theta}$ will be $\boldsymbol{s}_{abs} = (\boldsymbol{I} + \boldsymbol{\theta}\times)\boldsymbol{N}_d\ \boldsymbol{q}_a = \boldsymbol{N}_d\ \boldsymbol{q}_a + \boldsymbol{\theta} \times \boldsymbol{N}_d\ \boldsymbol{q}_a = \boldsymbol{N}_d\ \boldsymbol{q}_a$, because $\boldsymbol{\theta} \times \boldsymbol{N}_d\ \boldsymbol{q}_a$ is second order.

be enforced naturally by the PVW. Such an indirect enforcement of balance will often require a larger base to obtain an acceptable convergence of the motions. It will also impede a uniform, point wise convergence toward natural boundary conditions, possibly producing a local energy averaged convergence only, of the type Fourier series show at finite jump discontinuities. Therefore enforcing, even partially, the satisfaction of natural boundary conditions onto $\boldsymbol{N}_d(\boldsymbol{x})$ and $\boldsymbol{N}_\phi(\boldsymbol{x})$ will greatly enhance convergence, of stresses particularly.

- **How they do converge**
  To help in having it clear we illustrate qualitatively the kind of convergence obtainable through the PVW when natural (balance, Neuman) boundary conditions are not satisfied [2]. For that we look for an approximated torsional rotation response, $\theta(x,t)$, and internal torque distribution, $M_t(x,t)$, related to a beam: having a uniform torsional stiffness GJ, polar moment of inertia for unit length I, built in at $x(0)$, with a time stepped concentrated torque $M(t)$ applied at $x(L)$, assuming $\theta(x,0)=0$ and $\dot{\theta}(x,0)=0$ as initial conditions.



The presented results are based on the use of the exact vibration modes, normalized to unit generalized masses, with $M_t(x,t)$ recovered directly from $\theta(x,t)$. The related solution process can be found in many books of structural dynamics, e.g. [12], and need not be repeated here. Given the exact vibration modes frequencies ω_i:

$$\omega_i = (2i-1)\,\frac{\pi}{2L}\sqrt{\frac{GJ}{I}}$$

the related approximations are:

$$\theta(x,t) = \frac{8ML}{\pi^2 GJ}\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{(2i-1)^2}\sin\left((2i-1)\frac{\pi x}{2L}\right)(1-\cos\omega_i t)\,.$$

and, using the notation $\theta' = \frac{\partial\theta(x,t)}{\partial x}$:

$$M_t(x,t) = GJ\theta'(x,t) = \frac{4M}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{(2i-1)}\cos\left((2i-1)\frac{\pi x}{2L}\right)(1-\cos\omega_i t)$$

The vibration modes satisfy the boundary conditions, $\theta(0,t)=0$ and $\theta'(L,t)=0$. It is thus clear that their superposition will never meet the true natural boundary condition $M_t(L,t)=GJ\theta'(L,t)=M$.

Looking at the above solutions, for any assigned time t of interest, we can see that the rotation appears as a Fourier series of a time antisymmetric (sin based) periodic function of period $4L$, which interest us just for $0 \le x \le L$. So it is as complete as any of such series [1]. Its coefficients are alternating in sign and asymptotic to $\frac{1}{i^2}$, which is an indication of good (uniform) convergence toward a continuous function. So, despite

[2] Savvy and mathematically well inclined modern dynamicists may not like this lengthy example, on the base that using functional analysis concepts, [20], all of what it tells can be synthesized in a few lines.

violating a natural boundary condition, the rotation behaves well, not so for its derivatives though. In fact knowing a little about Fourier series and focusing onto $M_t(x,t)$, i.e. $\theta^{'}(x,t)$, it can be inferred that its behavior, alternating sign asymptotic to $\frac{1}{i}$, characterizes functions with finite jumps, the related point wise convergence being at the middle of the amplitude of the jump, accompanied by Gibbs type oscillations. So $M_t(x,t)$ converges point wise to 0 at L while simultaneously getting steeper and closer to M from the left. Consequently we are hinted a general qualitative picture on how, because of the use of the PVW, natural boundary conditions tend to be satisfied by an approximation that does not contain them. That would be enough to cast some light on the quality to impose to a complete displacement base. We dare nonetheless being a little more boring and notice that, after separating the two terms of the summation, i.e. those of the 1 and $\cos\omega_i t$, some training with elementary Fourier series allows to guess that the term of the 1: $\frac{4M}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{(2i-1)}\cos\left((2i-1)\frac{\pi x}{2L}\right)$ is nothing but the series of a time symmetric (cos based) periodic square wave of period $4L$ and amplitude M. So within $0 \leq x \leq L$ we can write:

$$M_t(x,t) = M - \frac{4M}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{(2i-1)}\cos\left((2i-1)\frac{\pi x}{2L}\right)\cos\omega_i t.$$

with which we can see that, while still having a somewhat slow convergence, the true natural boundary condition at L is now satisfied exactly. Then the problem becomes one of automatically putting "some training with elementary Fourier series" in the calculations. Such trained calculations are well known to structural dynamicists under the names: modes acceleration, inertia segregation, direct summation of (generalized) forces [2, 27]. Whatever the name the simple idea is the same for all of them:

- use any available, well behaving, motion approximation to calculate any motion dependent generalized force,
- apply to the structure the forces so obtained and those explicitly known,
- then solve it statically at each time instant of interest.

In our case that means loading the structure with a pure torque distribution combining M and the known continuous inertia couples:

$$\begin{aligned} &m_t(x,t) = M\ \delta(x-L) - I\ddot{\theta} = \\ &M\ \delta(x-L) - \frac{2M}{L}\sum_{i=1}^{\infty}(-1)^{i-1}\sin\left((2i-1)\frac{\pi x}{2L}\right)\cos\omega_i t. \end{aligned}$$

with the easiest name to pick up for our trained agent being the "summation of forces", so that:

$$M_t(x,t) = \int_x^{L^+} m_t(\eta,t)d\eta = M - \frac{4M}{\pi}\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{(2i-1)}\cos\left((2i-1)\frac{\pi x}{2L}\right)\cos\omega_i t.$$

showing that the "summation of forces" has studied what it had to of the Fourier series, likely so its friends: "modes acceleration" and "inertia segregation". We end by noting that an engineer sizing the beam at its free end will care little or none of the lack of uniform point wise convergence at L, but will readily infer that the sizing torque is the value being approached by the series at L from the left, happily using it for its design. That is what happens when using modal analyses within FE approximations.

Enlightened, hopefully, by the previous example we recall that notable, well known ways of defining $\boldsymbol{N}_d(\boldsymbol{x})$ and $\boldsymbol{N}_\phi(\boldsymbol{x})$ so to satisfy natural boundary conditions are the reduced order global solutions obtained by calculating reaction less inertia relieved displacements associated to an appropriately chosen load base. For attached shape functions producing non null statically determinate reactions the cancellation of their effect will come "by nature" through the PVW as any reaction must, compulsory, be totally forgotten and not included as an applied external forces.

Using such a motion representation in the PVW we obtain the following mass matrix:

$$\boldsymbol{M}_a = \begin{bmatrix} \int_V \boldsymbol{R}^T \boldsymbol{m}\ \boldsymbol{R}\, dV & \int_V \boldsymbol{R}^T \boldsymbol{m}\ \boldsymbol{N}_a\ dV \\ \int_V \boldsymbol{N}_a^T \boldsymbol{m}\ \boldsymbol{R}\, dV & \int_V \boldsymbol{N}_a^T \boldsymbol{m}\ \boldsymbol{N}_a\ dV \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_r & \boldsymbol{M}_{rq} \\ \boldsymbol{M}_{qr} & \boldsymbol{M}_q \end{bmatrix} \tag{11}$$

with $\boldsymbol{M}_{qr} = \boldsymbol{M}_{rq}^T$. From the expression of $\boldsymbol{M}_r$ we can see that by representing rigid motions with three pure translations and rotations we have the very same mass matrix layout:

$$\boldsymbol{M}_r = \begin{bmatrix} \text{Diag}\,\{M_b\} & \boldsymbol{S} \\ \boldsymbol{S}^T & \boldsymbol{J} \end{bmatrix} \tag{12}$$

of a rigid body of mass M_b, with the antisymmetric static unbalance matrix $\boldsymbol{S}$, being 0 when $\boldsymbol{x}_o$ is its center of mass, and the symmetric moment of inertia $\boldsymbol{J}$ being diagonal when the $\boldsymbol{x}$ axes are the principal ones.

Then, to approximate the internal virtual work, we define the following small strain

$$\boldsymbol{B} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \end{bmatrix} \tag{13}$$

so that, applying it to Eq. (9), we can write:

$$\boldsymbol{\varepsilon} = \boldsymbol{B}(\boldsymbol{x}) \begin{Bmatrix} \boldsymbol{r}_a(t) \\ \boldsymbol{q}_a(t) \end{Bmatrix} \tag{14}$$

Combining the above approximated strains with the constitutive equation:

$$\boldsymbol{\sigma} = \boldsymbol{D}_0(\boldsymbol{x})\boldsymbol{\varepsilon} + \boldsymbol{D}_1(\boldsymbol{x})\dot{\varepsilon} \tag{15}$$

we obtain the stiffness:

$$\boldsymbol{K}_a = \begin{bmatrix} 0 & 0 \\ 0 & \int_V \boldsymbol{B}^T \boldsymbol{D}_0\ \boldsymbol{B}\ dV \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{K}_q \end{bmatrix} \tag{16}$$

and damping:

$$\boldsymbol{C}_a = \begin{bmatrix} 0 & 0 \\ 0 & \int_V \boldsymbol{B}^T \boldsymbol{D}_1 \boldsymbol{B}\ dV \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{C}_q \end{bmatrix} \tag{17}$$

matrices. The previous derivation of $\boldsymbol{C}_a$ should be intended just as a plausible and expedite way to introduce some form of structural damping. Other, more complete, forms could have been devised as well, but they do not pertain to this tutorial. However derived the only thing that will count in the following is that $\boldsymbol{C}_q$ should be positive definite and capable of producing relatively small dissipations. Finally the generalized external load is:

$$\boldsymbol{Q}_a = \begin{Bmatrix} \int_V \boldsymbol{R}^T \begin{Bmatrix} \boldsymbol{f}_v \\ \boldsymbol{c}_v \end{Bmatrix} dV + \int_S \boldsymbol{R}^T \begin{Bmatrix} \boldsymbol{f}_s \\ \boldsymbol{c}_s \end{Bmatrix} dS \\ \int_V \boldsymbol{N}_a^T \begin{Bmatrix} \boldsymbol{f}_v \\ \boldsymbol{c}_v \end{Bmatrix} dV + \int_S \boldsymbol{N}_a^T \begin{Bmatrix} \boldsymbol{f}_s \\ \boldsymbol{c}_s \end{Bmatrix} dS \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{Q}_{ar} \\ \boldsymbol{Q}_{aq} \end{Bmatrix} \tag{18}$$

so that the response equation in attached axes is:

$$\begin{bmatrix} \boldsymbol{M}_r & \boldsymbol{M}_{rq} \\ \boldsymbol{M}_{qr} & \boldsymbol{M}_q \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{r}}_a \\ \ddot{\boldsymbol{q}}_a \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{C}_q \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{r}}_a \\ \dot{\boldsymbol{q}}_a \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{K}_q \end{bmatrix} \begin{Bmatrix} \boldsymbol{r}_a \\ \boldsymbol{q}_a \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{Q}_{ar} \\ \boldsymbol{Q}_{aq} \end{Bmatrix} \tag{19}$$

where only motion coordinates and generalized load vectors have the "a" suffix to mark their relation to an attached system. The mass, damping and stiffness matrices are not marked in the same way, because they will be used to define the related matrices for the mean axes also, without repeating any integration.

3. Getting to response equations in mean axes

If in place of $\boldsymbol{N}_a$ it was possible to choose a set $\boldsymbol{N}_m$ so to have:

$$(a): \quad \int_V \boldsymbol{R}^T \boldsymbol{m}\ \boldsymbol{N}_m\ dV = 0 \ (b): \quad \int_V \boldsymbol{N}_m^T\ \boldsymbol{m}\ \boldsymbol{R}\ dV = 0 \tag{20}$$

Then Eq. (11) shows that we would be led to decoupling rigid and deformable inertia forces. In fact the above equations are nothing but the definition of mean axes.

In such a view we have to notice that the direct derivation using assigned shape functions for a raw attached approximation of a statically determinate structure, i.e. Eq. (9), is a relatively easy task. On the contrary guessing a priori the shape functions $\boldsymbol{N}_m(\boldsymbol{x})$ satisfying Eq. (20) is a bit more difficult. In fact the excellence in mean axes are the vibration modes but they are not for free, a lot of calculations being required for their determination. We note also that the definition of mean axes implies that the resulting overall linear and angular momenta of the deformable structure, for small motions, remain the same as those of the rigid structure, so that there will be no inertia coupling between a floating mean reference frame and structural deformations. So, in turn, the center of mass of a deforming structure remains the same as that of its parent rigid body [9, 17]. Consequently the equations of motion of the reference frame will be the same as those of the related rigid body.

Being not as easy to assign $\boldsymbol{N}_m(\boldsymbol{x})$ as $\boldsymbol{N}_a(\boldsymbol{x})$ we exploit the latter to get the former. Therefore, since any linear combination of the elements of a complete base remains a base, we can define a new set of shape function $\boldsymbol{N}_m(\boldsymbol{x})$ using an appropriate linear combination of $\boldsymbol{N}_a(\boldsymbol{x})$ and $\boldsymbol{R}$, i.e.

$$\boldsymbol{N}_m(\boldsymbol{x}) = \boldsymbol{N}_a(\boldsymbol{x}) - \boldsymbol{R}(\boldsymbol{x})\ \boldsymbol{z} \tag{21}$$

Imposing that $\boldsymbol{N}_m(\boldsymbol{x})$ satisfies the mean axes definition (20)(a):

$$\begin{aligned} &\int_V \boldsymbol{R}^T \boldsymbol{m}\ \boldsymbol{N}_m\ dV = \int_V \boldsymbol{R}^T \boldsymbol{m}(\boldsymbol{N}_a - \boldsymbol{R}\ \boldsymbol{z})\ dV = \\ &\int_V \boldsymbol{R}^T \boldsymbol{m}\ \boldsymbol{N}_a\ dV - \int_V \boldsymbol{R}^T \boldsymbol{m}\ \boldsymbol{R}\ \ dV\ \boldsymbol{z} = \boldsymbol{M}_{rq} - \boldsymbol{M}_r\ \boldsymbol{z} = 0 \end{aligned} \tag{22}$$

we have:

$$\boldsymbol{z} = \boldsymbol{M}_r^{-1} \boldsymbol{M}_{rq} \tag{23}$$

so that:

$$\boldsymbol{N}_m(\boldsymbol{x}) = \boldsymbol{N}_a(\boldsymbol{x}) - \boldsymbol{R}(\boldsymbol{x}) \boldsymbol{M}_r^{-1} \boldsymbol{M}_{rq} \tag{24}$$

thus allowing to obtain the corresponding RA in mean axes:

$$\boldsymbol{s} = \begin{bmatrix} \boldsymbol{R}(\boldsymbol{x}) & (\boldsymbol{N}_a(\boldsymbol{x}) - \boldsymbol{R}(\boldsymbol{x})\ \boldsymbol{M}_r^{-1} \boldsymbol{M}_{rq}) \end{bmatrix} \begin{Bmatrix} \boldsymbol{r}_m \\ \boldsymbol{q}_m \end{Bmatrix}, \tag{25}$$

It might be of interest to recall that the above recovery of mean axes from any corresponding attached set is, algebraically speaking, nothing but the application of a generalized Gram-Schmidt subspace orthogonalization procedure.

Rewriting the above equation as

$$\boldsymbol{s} = \begin{bmatrix} \boldsymbol{R}(\boldsymbol{x}) & \boldsymbol{N}_a(\boldsymbol{x}) \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{M}_r^{-1}\boldsymbol{M}_{rq} \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{r}_m \\ \boldsymbol{q}_m \end{Bmatrix}, \tag{26}$$

we can infer the following transformation of the free amplitude coordinates only:

$$\begin{Bmatrix} \boldsymbol{r}_a \\ \boldsymbol{q}_a \end{Bmatrix} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{M}_r^{-1}\boldsymbol{M}_{rq} \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{r}_m \\ \boldsymbol{q}_m \end{Bmatrix} \tag{27}$$

and its inverse

$$\begin{Bmatrix} \boldsymbol{r}_m \\ \boldsymbol{q}_m \end{Bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{M}_r^{-1}\boldsymbol{M}_{rq} \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{r}_a \\ \boldsymbol{q}_a \end{Bmatrix} \tag{28}$$

It is remarked that, being $\boldsymbol{q}_a = \boldsymbol{q}_m$ true always, the difference between the motions in attached and mean axes is just in the related rigid parts. The possibility of reading Eq. (25) as Eq. (26) makes it possible to change anything from attached to mean axes by using just the standard energy preserving, purely algebraic, transformation, without repeating any integration related to the use of Eq. (25), in place of Eq. (9), into the PVW. [3] So we end with the following mean axes matrices and load:

$$\boldsymbol{M}_m = \begin{bmatrix} \boldsymbol{M}_r & 0 \\ 0 & \boldsymbol{M}_q - \boldsymbol{M}_{rq}^T\ \boldsymbol{M}_r^{-1}\ \boldsymbol{M}_{rq} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_r & 0 \\ 0 & \boldsymbol{M}_{mq} \end{bmatrix} \tag{29}$$

$$\boldsymbol{C}_m = \boldsymbol{C}_a = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{C}_q \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{C}_{mq} \end{bmatrix} \tag{30}$$

$$\boldsymbol{K}_m = \boldsymbol{K}_a = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{K}_q \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{K}_{mq} \end{bmatrix} \tag{31}$$

$$\begin{Bmatrix} \boldsymbol{Q}_{mr} \\ \boldsymbol{Q}_{mq} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{I} & 0 \\ -\boldsymbol{M}_{rq}^T\ \boldsymbol{M}_r^{-1} & \boldsymbol{I} \end{bmatrix}^T \begin{Bmatrix} \boldsymbol{Q}_{ar} \\ \boldsymbol{Q}_{aq} \end{Bmatrix} \tag{32}$$

so that the equation of motion in mean axes is:

$$\begin{bmatrix} \boldsymbol{M}_r & 0 \\ 0 & \boldsymbol{M}_{mq} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{r}}_m \\ \ddot{\boldsymbol{q}}_m \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{C}_q \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{r}}_m \\ \dot{\boldsymbol{q}}_m \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{K}_q \end{bmatrix} \begin{Bmatrix} \boldsymbol{r}_m \\ \boldsymbol{q}_m \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{Q}_{mr} \\ \boldsymbol{Q}_{mq} \end{Bmatrix} \tag{33}$$

What above well shows the full uncoupling of the frame equation of motion from the deformable part. So, calling s any eigenvalue, the deformable vibration modes are given just by:

$$\boldsymbol{K}_q\ \boldsymbol{q}_m = (-s^2\boldsymbol{M}_{mq} - s\ \boldsymbol{C}_q)\ \boldsymbol{q}_m \tag{34}$$

$$\boldsymbol{r}_m = 0 \qquad \boldsymbol{s} = (\boldsymbol{N}_a(\boldsymbol{x}) - \boldsymbol{R}(\boldsymbol{x})\ \boldsymbol{M}_r^{-1}\boldsymbol{M}_{rq})\boldsymbol{q}_m \tag{35}$$

Consequently their determination stands fully alone now. Therefore the introductory promise that the use of the mean axes would have provided a lean and direct way to separate the rigid and deformable vibration modes has been swiftly maintained.

We want to remark here that the already mentioned important mean axes property of the equation of the dynamics of the reference frame remaining the same as that of the rigid body, adds a related important implication in that any imposed constant acceleration field $\ddot{\boldsymbol{r}}_m$, e.g. gravity, does not contribute to any component of $\boldsymbol{Q}_{mq}$[17]. It is also interesting to see that the new generalized load vector exciting deformable motions, $\boldsymbol{Q}_{mq} = \boldsymbol{Q}_{aq} - \boldsymbol{M}_{rq}^T\ \boldsymbol{M}_r^{-1}\ \boldsymbol{Q}_{ar}$, corresponds

[3] Let us remind the well known work/energy preserving transformations [19], whereas given any generalized coordinate transformation matrix T the corresponding transformations of the matrices $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ and of the load $\boldsymbol{Q}$ are: $\boldsymbol{T}^T\ \boldsymbol{M}\ \boldsymbol{T}$, $\boldsymbol{T}^T\ \boldsymbol{C}\ \boldsymbol{T}$, $\boldsymbol{T}^T\ \boldsymbol{K}\ \boldsymbol{T}$ and $\boldsymbol{T}^T\ \boldsymbol{Q}$. It is noticed that, while a right product combines the columns so to satisfy an imposed coordinate transformation, the left product implies a combination of rows, so of the components of the related generalized forces. An operation which is often called a generalized projection.

to an inertia relieved load. In fact $\boldsymbol{Q}_{ar}$ is the resultant force-moment, $\boldsymbol{M}_r^{-1}\,\boldsymbol{Q}_{ar}$ the instantaneous acceleration, so that $-\boldsymbol{M}_{rq}^T\,\boldsymbol{M}_r^{-1}\,\boldsymbol{Q}_{ar}$ is the corresponding inertia relief. The same interpretation can be applied to the change of the mass matrix as the term $-\boldsymbol{M}_{rq}^T\,\boldsymbol{M}_r^{-1}\,\boldsymbol{M}_{rq}$ is nothing but the inertia relief of the coupling inertia forces $\boldsymbol{M}_{rq}\,\ddot{\boldsymbol{q}}_a$. Such an inertia relief plays an important part since it alone is capable of uncoupling the frame and deformable motions, as it should be easily seen by applying to Eq. (19) the left transformation (projection) of Eq. (27) only, obtaining:

$$\begin{bmatrix}\boldsymbol{M}_r & \boldsymbol{M}_{rq}\\ 0 & \boldsymbol{M}_{mq}\end{bmatrix}\begin{Bmatrix}\ddot{\boldsymbol{r}}_a\\ \ddot{\boldsymbol{q}}_a\end{Bmatrix}+\begin{bmatrix}0 & 0\\ 0 & \boldsymbol{C}_q\end{bmatrix}\begin{Bmatrix}\dot{\boldsymbol{r}}_a\\ \dot{\boldsymbol{q}}_a\end{Bmatrix}+\begin{bmatrix}0 & 0\\ 0 & \boldsymbol{K}_q\end{bmatrix}\begin{Bmatrix}\boldsymbol{r}_a\\ \boldsymbol{q}_a\end{Bmatrix}=\begin{Bmatrix}\boldsymbol{Q}_{ar}\\ \boldsymbol{Q}_{mq}\end{Bmatrix} \tag{36}$$

Recalling that the PVW allows to combine any compatible virtual movement with any set of internal-external dynamically balanced generalized forces, it is possible to verify that the previous results corresponds to using Eq. (25) for the virtual variations and Eq. (9) for the displacements associated to any actual motion. The other way around, i.e. applying only the right transformation of Eq. (27), or exchanging the virtual-actual motions used for getting Eq. (36), will give:

$$\begin{bmatrix}\boldsymbol{M}_r & 0\\ \boldsymbol{M}_{qr} & \boldsymbol{M}_{mq}\end{bmatrix}\begin{Bmatrix}\ddot{\boldsymbol{r}}_m\\ \ddot{\boldsymbol{q}}_m\end{Bmatrix}+\begin{bmatrix}0 & 0\\ 0 & \boldsymbol{C}_q\end{bmatrix}\begin{Bmatrix}\dot{\boldsymbol{r}}_m\\ \dot{\boldsymbol{q}}_m\end{Bmatrix}+\begin{bmatrix}0 & 0\\ 0 & \boldsymbol{K}_q\end{bmatrix}\begin{Bmatrix}\boldsymbol{r}_m\\ \boldsymbol{q}_m\end{Bmatrix}=\begin{Bmatrix}\boldsymbol{Q}_{ar}\\ \boldsymbol{Q}_{aq}\end{Bmatrix} \tag{37}$$

which is somewhat of less interest, since it keeps the deformable part interacting with rigid motions. Considering that $\boldsymbol{q}_a=\boldsymbol{q}_m$ Eqs. (33), (36) and (37) provide the same deformable modes problem as Eq. (34), the only difference being confined to using either $\boldsymbol{r}_m$ or $\boldsymbol{r}_a$ in the recovery of the overall eigenfunction of any mode. It is nonetheless a fact that by applying just the left-right transformations we loose symmetry for the whole set of equations of motion and so the complete uncoupling warranted by mean axes [4]. The mean axes formulation of the equations of motion shows that the related representation of the small movements of a deformable body can be seen just as a matter of recovering a reference frame within a uniquely positioned free structure, by using the transformations Eqs. (27), (28). either a priori, so producing Eq. (33), or, as a kind of post processing, a posteriori, of any solution of Eq. (19). This fact seems to support the idea that there is no use in adopting mean axes but, even so, such an attitude is not correct for linear deformable structural systems. In fact mean axes can have a profound effect also on the modeling of forces depending upon the motion of the structure. That is the case for the linear(ized) aerodynamic approximations often used in aeroelasticity and flight mechanics, for which it is possible to produce aerodynamic corrections taking into account a deformable structure, without changing the model structure of the corresponding rigid body, only if mean axes are adopted [11, 21].

4. Specialization to Finite Elements

The specialization of the above ideas to a FE approximation of a free structure, written as:

$$\boldsymbol{M}_{fe}\,\ddot{\boldsymbol{u}}+\boldsymbol{C}_{fe}\,\ddot{\boldsymbol{u}}+\boldsymbol{K}_{fe}\,\boldsymbol{u}=\boldsymbol{P}_{fe} \tag{38}$$

is done in a straightforward manner by applying what developed for Eqs. (19) and (33) directly to the FE nodal variables, without caring of the underlying FE shape functions and integrals associated to the PVW discretization, which are embedded in $\boldsymbol{M}_{fe}$, $\boldsymbol{K}_{fe}$ and $\boldsymbol{P}_{fe}$ already. It should be taken into account that, to avoid spurious connections to the ground, the structural damping matrix $\boldsymbol{C}_{fe}$

[4] The coincidence of the PVW with a Galerkin weighted residual approach should be well known. Nowadays Eq. (33) would be related to a Bubnov-Galerkin approximation and Eqs. (36), (37) to a Petrov-Galerkin one [20]

must be as definite as $\boldsymbol{K}_{fe}$. So we define at first the nodal rigid motions and attached deformable base coordinates, i.e. the equivalent of (9) as:

$$\boldsymbol{u} = \begin{bmatrix}\boldsymbol{R} & \boldsymbol{U}_a\end{bmatrix}\begin{Bmatrix}\boldsymbol{r}_a \\ \boldsymbol{u}_a\end{Bmatrix} \tag{39}$$

in which the attached deformable base, $\boldsymbol{U}_a$, corresponds to the nodal motions of a set of free nodes related to any statically determinate structure. As it was remarked for $\boldsymbol{N}_d$ and $\boldsymbol{N}_\phi$ we recall that, being small, $\boldsymbol{U}_a$ can be indifferently seen as both absolute or relative to the free body frame motions. So $\boldsymbol{U}_a$ is made by columns having a number of rows equal to the whole number of FE degrees of freedom, including those of the statically determinate constraints, with just a 1 in the position corresponding to their free degree of freedom, counted again with the inclusion of those of the statically determinate constraints. To have it clearer we notice that any pre multiplication of a matrix/vector by $\boldsymbol{U}_a^T$ will reduce their number of rows to that of the free degrees of freedom with the extraction of each row corresponding to the single number one in each column of $\boldsymbol{U}_a$. In the same way any post multiplication of a matrix by $\boldsymbol{U}_a$ will reduce its number of columns to that of the free degrees of freedom with the extraction of the columns corresponding to the ones in each columns of $\boldsymbol{U}_a$. Taking for example the matrix $\boldsymbol{K}_{fe}$ we can see that the application of the mentioned pre-post multiplications will be tantamount to making the structure statically determinate by eliminating the degrees of freedom of the related constraints. On the contrary a pre multiplication of a matrix/vector by $\boldsymbol{U}_a$ will expand its rows back to the number of all of the original FE nodal degrees of freedom, while a post multiplications of a matrix by $\boldsymbol{U}_a^T$ will do the very same for its columns.

We can then obtain the following attached mass/damping/stiffness matrices and load:

$$\boldsymbol{M}_a = \begin{bmatrix}\boldsymbol{R}^T\,\boldsymbol{M}_{fe}\,\boldsymbol{R} & \boldsymbol{R}^T\,\boldsymbol{M}_{fe}\,\boldsymbol{U}_a \\ \boldsymbol{U}_a^T\,\boldsymbol{M}_{fe}\,\boldsymbol{R} & \boldsymbol{U}_a^T\,\boldsymbol{M}_{fe}\,\boldsymbol{U}_a\end{bmatrix} = \begin{bmatrix}\boldsymbol{M}_r & \boldsymbol{M}_{ru} \\ \boldsymbol{M}_{ur} & \boldsymbol{M}_u\end{bmatrix} \tag{40}$$

with $\boldsymbol{M}_{ur} = \boldsymbol{M}_{ru}^T$.

$$\boldsymbol{C}_a = \begin{bmatrix}\boldsymbol{R}^T\,\boldsymbol{C}_{fe}\,\boldsymbol{R} & \boldsymbol{R}^T\,\boldsymbol{C}_{fe}\,\boldsymbol{U}_a \\ \boldsymbol{U}_a^T\,\boldsymbol{C}_{fe}\,\boldsymbol{R} & \boldsymbol{U}_a^T\,\boldsymbol{C}_{fe}\,\boldsymbol{U}_a\end{bmatrix} = \begin{bmatrix}0 & 0 \\ 0 & \boldsymbol{U}_a^T\,\boldsymbol{C}_{fe}\,\boldsymbol{U}_a\end{bmatrix} = \begin{bmatrix}0 & 0 \\ 0 & \boldsymbol{C}_u\end{bmatrix} \tag{41}$$

$$\boldsymbol{K}_a = \begin{bmatrix}\boldsymbol{R}^T\,\boldsymbol{K}_{fe}\,\boldsymbol{R} & \boldsymbol{R}^T\,\boldsymbol{K}_{fe}\,\boldsymbol{U}_a \\ \boldsymbol{U}_a^T\,\boldsymbol{K}_{fe}\,\boldsymbol{R} & \boldsymbol{U}_a^T\,\boldsymbol{K}_{fe}\,\boldsymbol{U}_a\end{bmatrix} = \begin{bmatrix}0 & 0 \\ 0 & \boldsymbol{U}_a^T\,\boldsymbol{K}_{fe}\,\boldsymbol{U}_a\end{bmatrix} = \begin{bmatrix}0 & 0 \\ 0 & \boldsymbol{K}_u\end{bmatrix} \tag{42}$$

$$\boldsymbol{P}_a = \begin{bmatrix}\boldsymbol{R}^T \\ \boldsymbol{U}_a^T\end{bmatrix}\boldsymbol{P}_{fe} = \begin{Bmatrix}\boldsymbol{P}_{ar} \\ \boldsymbol{P}_{au}\end{Bmatrix} \tag{43}$$

whose corresponding equation of motion is:

$$\begin{bmatrix}\boldsymbol{M}_r & \boldsymbol{M}_{ru} \\ \boldsymbol{M}_{ur} & \boldsymbol{M}_u\end{bmatrix}\begin{Bmatrix}\ddot{\boldsymbol{r}}_a \\ \ddot{\boldsymbol{u}}_a\end{Bmatrix} + \begin{bmatrix}0 & 0 \\ 0 & \boldsymbol{C}_u\end{bmatrix}\begin{Bmatrix}\dot{\boldsymbol{r}}_a \\ \dot{\boldsymbol{u}}_a\end{Bmatrix} + \begin{bmatrix}0 & 0 \\ 0 & \boldsymbol{K}_u\end{bmatrix}\begin{Bmatrix}\boldsymbol{r}_a \\ \boldsymbol{u}_a\end{Bmatrix} = \begin{Bmatrix}\boldsymbol{P}_{ar} \\ \boldsymbol{P}_{au}\end{Bmatrix} \tag{44}$$

i.e. the same as Eq. (19), apart of the use of $\boldsymbol{u}$ in place of $\boldsymbol{q}$ to recall their FE origin. It can be seen that attached axes mostly maintain the sparsity of Eq. (38), the only additional coupling added pertains to the mass matrix only and is related to the introduction of a small number of active rows-columns, i.e. $\boldsymbol{M}_{ru}$ and $\boldsymbol{M}_{ur}$, in the same number as the rigid modes.

To obtain a mean axes representation we can repeat the same procedure already used for the continuous RA, applying it directly to the already discretized FE form. Therefore a combination of the attached nodal degrees of freedom and rigid modes is defined

$$\boldsymbol{U}_m = \boldsymbol{U}_a - \boldsymbol{R}\,\boldsymbol{z} \tag{45}$$

and used for the mass orthogonalization required to decouple deformable and rigid modes

$$\boldsymbol{R}^T \boldsymbol{M}_{fe}\, \boldsymbol{U}_m = \boldsymbol{R}^T \boldsymbol{M}_{fe}\, (\boldsymbol{U}_a - \boldsymbol{R}\, \boldsymbol{z}) = \boldsymbol{M}_{ru} - \boldsymbol{M}_r\, \boldsymbol{z} = 0 \tag{46}$$

whose solution

$$\boldsymbol{z} = \boldsymbol{M}_r^{-1} \boldsymbol{M}_{ru} \tag{47}$$

allows to define a corresponding mean axes nodal base

$$\boldsymbol{U}_m = (\boldsymbol{I} - \boldsymbol{R}\, \boldsymbol{M}_r^{-1}\, \boldsymbol{R}^T\, \boldsymbol{M}_{fe})\boldsymbol{U}_a = \boldsymbol{T}\, \boldsymbol{U}_a \tag{48}$$

$\boldsymbol{T} = \boldsymbol{I} - \boldsymbol{R}\, \boldsymbol{M}_r^{-1}\, \boldsymbol{R}^T\, \boldsymbol{M}_{fe}$, is an idempotent matrix whose expression can be compacted, without loosing sparsity, by taking the Cholesky factorization: $\boldsymbol{M}_r = \boldsymbol{L}_r\, \boldsymbol{L}_r^T$, so that, defining the rigid modes normalized to unit mass $\boldsymbol{R}_n = \boldsymbol{R}\, \boldsymbol{L}_r^{-T}$ and $\boldsymbol{M}_{md} = \boldsymbol{M}_{fe}\, \boldsymbol{R}_n$, we can write: $\boldsymbol{T} = \boldsymbol{I} - \boldsymbol{R}_n\, \boldsymbol{M}_{md}^T$. Eq. (48) provides the following absolute mean nodal motions representation, i.e. the equivalent of Eq. (25),

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{T}\boldsymbol{U}_a \end{bmatrix} \begin{Bmatrix} \boldsymbol{r}_m \\ \boldsymbol{u}_m \end{Bmatrix} \tag{49}$$

Rewriting it as:

$$\boldsymbol{u} = \begin{Bmatrix} \boldsymbol{R} & \boldsymbol{U}_a \end{Bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{M}_r^{-1}\boldsymbol{M}_{ru} \\ 0 & \boldsymbol{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{r}_m \\ \boldsymbol{u}_m \end{Bmatrix} \tag{50}$$

it should be easy to infer from it Eqs. (27) and (28), with only the already mentioned care of using $\boldsymbol{u}$ in place of $\boldsymbol{q}$. Using the above transformations, as done before, we can derive the mean axes nodal mass matrices:

$$\boldsymbol{M}_m = \begin{bmatrix} \boldsymbol{M}_r & 0 \\ 0 & \boldsymbol{U}_a^T\, \boldsymbol{T}^T \boldsymbol{M}_{fe}\, \boldsymbol{T}\, \boldsymbol{U}_a \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}_r & 0 \\ 0 & \boldsymbol{M}_{mu} \end{bmatrix} \tag{51}$$

damping

$$\boldsymbol{C}_m = \boldsymbol{C}_a = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{C}_u \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{U}_a^T\, \boldsymbol{C}_{fe}\, \boldsymbol{U}_a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{C}_{mu} \end{bmatrix} \tag{52}$$

stiffness

$$\boldsymbol{K}_m = \boldsymbol{K}_a = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{K}_u \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{U}_a^T\, \boldsymbol{K}_{fe}\, \boldsymbol{U}_a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{K}_{mu} \end{bmatrix} \tag{53}$$

and load

$$\boldsymbol{P}_m = \begin{bmatrix} \boldsymbol{R}^T \\ \boldsymbol{U}_a^T\, \boldsymbol{T}^T \end{bmatrix} \begin{Bmatrix} \boldsymbol{P}_{ar} \\ \boldsymbol{P}_{au} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{P}_{mr} \\ \boldsymbol{P}_{mu} \end{Bmatrix} \tag{54}$$

with their corresponding fully decoupled equation of motion

$$\begin{bmatrix} \boldsymbol{M}_r & 0 \\ 0 & \boldsymbol{M}_{mu} \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{r}}_m \\ \ddot{\boldsymbol{u}}_m \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{C}_u \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{r}}_m \\ \dot{\boldsymbol{u}}_m \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \boldsymbol{K}_u \end{bmatrix} \begin{Bmatrix} \boldsymbol{r}_m \\ \boldsymbol{u}_m \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{P}_{mr} \\ \boldsymbol{P}_{mu} \end{Bmatrix} \tag{55}$$

i.e. the same as Eq. (33). Before presenting a few uses of a finite elements mean axes formulation we introduce a compact form of $\boldsymbol{M}_{mu}$, which will be useful in preserving sparse operations. To such an end, using the previously defined $\boldsymbol{M}_{md}$ we write:

$$\begin{aligned} \boldsymbol{M}_{mu} \quad &= \quad \boldsymbol{U}_a^T(\boldsymbol{M}_{fe} - \boldsymbol{M}_{fe}\, \boldsymbol{R}\, \boldsymbol{M}_r^{-1}\, \boldsymbol{R}^T\, \boldsymbol{M}_{fe})\boldsymbol{U}_a = \\ & \quad\;\; \boldsymbol{U}_a^T(\boldsymbol{M}_{fe} - \boldsymbol{M}_{md}\, \boldsymbol{M}_{md}^T)\boldsymbol{U}_a \end{aligned} \tag{56}$$

It is then possible to see that, as for the attached case, the mean axes touch the sparsity of the mass matrix only. This time however in a much heavier way since the sparsity of the mass matrix pertaining to the deformable part is destroyed completely. As it will be seen in the following it is still possible to carry out many matrix operations on the mass matrix in an effective way by working with the above version Eq. (56). Moreover, it should be noticed that what we said about inertia relieved generalized loads and the use of left-right transformations only applies here too. Even if we will not touch the subject of the direct integration of Eq. (55), to be found in a companion tutorial, we like to remark that it is possible to operate in a sparsity efficient way also when solving the typical linear system, with a coefficient matrix of the type $(\alpha\ \boldsymbol{M}_{fe} + \beta\ \boldsymbol{C}_{fe} + \gamma\ \boldsymbol{K}_{fe} - \alpha\ \boldsymbol{M}_{md}\ \boldsymbol{M}_{md}^T)$, associated to the many implicit methods used for direct integration in structural analysis [5].

It might now be of interest to transform the second of Eqs. (55) into a flexibility form by explicitly solving it for $\boldsymbol{u}_m$ as function of inertia and external forces. To that end we project, i.e. left transform, the FE equations to mean axes while transforming the nodal displacements to mean axes only for the elastic term so that we can write:

$$\boldsymbol{K}_u\ \boldsymbol{u}_m = \boldsymbol{U}_a^T\ \boldsymbol{T}^T\ (\boldsymbol{P}_{fe} - \boldsymbol{M}_{fe}\ddot{\boldsymbol{u}} - \boldsymbol{C}_{fe}\dot{\boldsymbol{u}}) \tag{57}$$

Eq. (57) is readily usable for calculating the above mentioned free-free balanced conditions, including residualized dynamic responses, i.e. when $\ddot{\boldsymbol{u}} = \boldsymbol{B}\ \ddot{\boldsymbol{q}}$ and $\dot{\boldsymbol{u}} = \boldsymbol{B}\ \dot{\boldsymbol{q}}$, with $\boldsymbol{B}$ being a reduced response base associated to the free coordinates $\boldsymbol{q}$, whose time history has been obtained using a reduced model. Such a residualization is called a static inertia, with damping, relief when $\boldsymbol{B} = \boldsymbol{R}$ and modes acceleration when $\boldsymbol{B}$ is composed of a set of generic global deformable modes. We then call $\boldsymbol{F}_a = \boldsymbol{K}_u^{-1}$ the attached flexibility matrix and put the above into a flexibility form by explicitly solving it for $\boldsymbol{u}_m$:

$$\boldsymbol{u}_m = \boldsymbol{F}_a\ \boldsymbol{U}_a^T\ \boldsymbol{T}^T(\boldsymbol{P}_{fe} - \boldsymbol{M}_{fe}\ddot{\boldsymbol{u}} - \boldsymbol{C}_{fe}\dot{\boldsymbol{u}}) \tag{58}$$

so that after substituting Eq. (58) into Eq. (49), defining $\boldsymbol{F}_{ua} = \boldsymbol{U}_a\ \boldsymbol{F}_a\ \boldsymbol{U}_a^T$, the absolute attached generalized flexibility matrix, and $\boldsymbol{F}_{um} = \boldsymbol{T}\ \boldsymbol{F}_{ua}\boldsymbol{T}^T$, the absolute mean generalized flexibility matrix, we have:

$$\begin{aligned} \boldsymbol{u} \quad &= \quad \boldsymbol{R}\ \boldsymbol{r}_m + \boldsymbol{T}\ \boldsymbol{F}_{ua}\boldsymbol{T}^T\ (\boldsymbol{P}_{fe} - \boldsymbol{M}_{fe}\ \ddot{\boldsymbol{u}} - \boldsymbol{C}_{fe}\dot{\boldsymbol{u}}) = \\ &\quad\ \ \boldsymbol{R}\ \boldsymbol{r}_m + \boldsymbol{F}_{um}\ (\boldsymbol{P}_{fe} - \boldsymbol{M}_{fe}\ \ddot{\boldsymbol{u}} - \boldsymbol{C}_{fe}\dot{\boldsymbol{u}}) \end{aligned} \tag{59}$$

All of the expressions using the above flexibility matrices are of utmost help in synthesizing and manipulating the related formulas but are never used directly as such in any large numerical calculations. In practice the previous formula just suggests us what to do for solving a free structures in mean axes, for whatever load. In fact we can expand back its synthesized structure and reread it as:

- preserve sparsity as much as possible by factorizing once for all the statically determinate stiffness matrix of the left hand side of Eq. (57), i.e. $\boldsymbol{K}_u = \boldsymbol{L}_k\ \boldsymbol{L}_k^T$;
- inertia relieve whatever load $\boldsymbol{P}$ using $\boldsymbol{U}_a^T(\boldsymbol{P} - \boldsymbol{M}_{md}(\boldsymbol{R}_n^T\boldsymbol{P}))$, strictly as written, i.e. the right hand side of Eq. (57);
- solve the statically determinate Eq. (57) for $\boldsymbol{u}_m$, using forward-backward substitutions based on the triangular factor $\boldsymbol{L}_k$;

[5] Using any of the Woodbury type identities (http://matrixcookbook.com), it can be proven that the solution of a linear system of equations of the type $(\boldsymbol{A} - \boldsymbol{B}\ \boldsymbol{B}^T)\ \boldsymbol{x} = \boldsymbol{b}$, with $\boldsymbol{A}$ symmetric, positive definite and sparse and $\boldsymbol{B}$ having far less columns than the non zero terms in any row of the Cholesky factor $\boldsymbol{L}$ of $\boldsymbol{A}$, is given by: $\boldsymbol{x} = \boldsymbol{L}^{-T}(\boldsymbol{L}^{-1}\boldsymbol{b}) + \boldsymbol{C}(\boldsymbol{C}^T\boldsymbol{b})$. The matrix $\boldsymbol{C}$ can be obtained by simply applying forward-backward substitutions of $\boldsymbol{L}$ to $\boldsymbol{B}$ and has the very same structure as $\boldsymbol{B}$. So the solution cost will be mostly confined to the forward-backward steps $\boldsymbol{L}^{-T}(\boldsymbol{L}^{-1}\boldsymbol{b})$, i.e. very much the same as solving the sparse system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ only.

- recover a full solution to mean axes: $\boldsymbol{u}_t = \boldsymbol{U}_a \boldsymbol{u}_m$, $\boldsymbol{u} = \boldsymbol{u}_t - \boldsymbol{R}_n(\boldsymbol{M}_{md}^T \boldsymbol{u}_t)$, i.e. pre multiply the just obtained $\boldsymbol{u}_m$ by $\boldsymbol{T}\ \boldsymbol{U}_a$; then add any given rigid motion, $\boldsymbol{R}\ \boldsymbol{r}_m$, if needed.

Further numerical ways to do it efficiently will be explained later on, in connection to block power iterations.

The above mean axes approach applied to the deformable part only, i.e. with $\boldsymbol{r}_m = 0$, is the same as that introduced in an anticipatory paper on the calculation of deformable vibration modes only, dating back to 1955 [8]. Such a scheme, as reprised in [10, 12], leads to determining a pseudo inverse $\boldsymbol{G}$ of $\boldsymbol{K}_{fe}$, there called generalized matrix of influence coefficients, written as $\boldsymbol{K}_{fe}\ \boldsymbol{G} = \boldsymbol{A}^T$, where $\boldsymbol{A}$ is nothing but our $\boldsymbol{T}$. The resulting $\boldsymbol{G}$ turns out to be:

$$\boldsymbol{G}_{iso} = \begin{bmatrix} \boldsymbol{K}_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \boldsymbol{G} = \boldsymbol{A}\ \boldsymbol{G}_{iso}\ \boldsymbol{A}^T \tag{60}$$

$\boldsymbol{K}_{11}$ being a statically determinate partition of $\boldsymbol{K}_{fe}$, i.e. the same as $\boldsymbol{K}_u$. It is then easy to see that $\boldsymbol{G}$ is nothing but our $\boldsymbol{F}_{um}$. In fact, after recalling the definition of $\boldsymbol{U}_a$ a simple inspection shows that $\boldsymbol{G}_{iso} = \boldsymbol{F}_{ua}$.

Eq. (59) can be specialized to calculate the deformable only, $\boldsymbol{r}_m = 0$, static mean solutions related to inertia relieved loads only:

$$\boldsymbol{u} = \boldsymbol{F}_{um}\ \boldsymbol{P}_{fe} \tag{61}$$

and those related to static mean non self balanced loads:

$$\boldsymbol{u} = \boldsymbol{T}\ \boldsymbol{F}_{ua}\ \boldsymbol{P}_{fe} \tag{62}$$

The above expressions are useful for the determination of reduced models with a static displacement base, i.e. a linear combination of base loads given by: $\boldsymbol{P}_{fe} = \boldsymbol{L}_{fe}\ \boldsymbol{l}$, appended to vibration modes so to enhance dynamic response analyses, e.g. to better recover internal loads associated to high concentrated masses and forces-moments, without resorting to an a posteriori response recovery using modes acceleration [2, 27]. Solving with any of the above static solutions one ends with an attached-mean displacements base $\boldsymbol{U}_l\ \boldsymbol{l}$, so that the load base amplitudes $\boldsymbol{l}$ of $\boldsymbol{L}_{fe}$ become a set of new generalized free coordinates. In such a way one could even substitute any adequately complete displacement base with a complete load base, i.e. one capable of well representing any inertia and external loads combination.

We will now use the deformable equation in mean axes to set forth the corresponding deformable vibration only eigenproblem, showing how to use it in a sparsity preserving mode to maintain numerical efficiency in the power iterations. To such an end we rewrite the related damped and undamped eigenproblems as:

- $\boldsymbol{C}_{fe} \neq 0$, $s = \omega(\xi \pm j\ \sqrt{1-\xi^2})$, ξ being the damping factor:
  $\boldsymbol{K}_u\ \boldsymbol{u}_m = -s\boldsymbol{U}_a^T\ ((\boldsymbol{M}_{fe} - \boldsymbol{M}_{md}\ \boldsymbol{M}_{md}^T)\boldsymbol{U}_a\ \hat{\boldsymbol{u}}_m + s\boldsymbol{C}_{fe}\boldsymbol{U}_a\ \boldsymbol{u}_m)$
  $\hat{\boldsymbol{u}}_m = \lambda\ \boldsymbol{u}_m$
- $\boldsymbol{C}_{fe} = 0$:
  $\boldsymbol{K}_u\ \boldsymbol{u}_m = \omega^2\boldsymbol{U}_a^T\ ((\boldsymbol{M}_{fe} - \boldsymbol{M}_{md}\ \boldsymbol{M}_{md}^T)\boldsymbol{U}_a\ \boldsymbol{u}_m$

which are kept in expanded form because that is the way to use them in sparsity preserving power iterations. The related inverse power iterations will be:

- $\boldsymbol{C}_{fe} \neq 0$:

$$\begin{aligned} \boldsymbol{K}_u\ \boldsymbol{U}_m^{k+1} &= -\boldsymbol{U}_a^T((\boldsymbol{M}_{fe} - \boldsymbol{M}_{md}\ \boldsymbol{M}_{md}^T)\boldsymbol{U}_a\ \hat{\boldsymbol{U}}_m^k + \boldsymbol{C}_{fe}\boldsymbol{U}_a\ \boldsymbol{U}_m^k) \\ \hat{\boldsymbol{U}}_m^{k+1} &= \boldsymbol{U}_m^k \end{aligned} \tag{63}$$

- $\boldsymbol{C}_{fe} = 0$:
$$\boldsymbol{K}_u \, \boldsymbol{U}_m^{k+1} = \boldsymbol{U}_a^T(\boldsymbol{M}_{fe} - \boldsymbol{M}_{md} \, \boldsymbol{M}_{md}^T)\boldsymbol{U}_a \, \boldsymbol{U}_m^k \tag{64}$$

Since power iterations are mostly enhanced by combining them with the reduced order projections obtained according to a problem dependent Rayleigh quotient we report for completeness the symmetric ones associate to the above iterations:

$$s_R = \frac{\hat{\boldsymbol{u}}_m^T \boldsymbol{M}_{mu} \hat{\boldsymbol{u}}_m - \boldsymbol{u}_m^T \boldsymbol{K}_u \boldsymbol{u}_m}{2\hat{\boldsymbol{u}}_m^T \boldsymbol{M}_{mu} \, \boldsymbol{u}_m + \boldsymbol{u}_m^T \boldsymbol{C}_u \boldsymbol{u}_m} \qquad \omega_R^2 = \frac{\boldsymbol{u}_m^T \boldsymbol{K}_u \boldsymbol{u}_m}{\boldsymbol{u}_m \boldsymbol{M}_{mu} \, \boldsymbol{u}_m} \tag{65}$$

Omitting the iteration indexes for ease of notation, an actual implementation of the above iterations will be carried out with the following series of calculation for the damped case, the undamped case being much the same:

1. $\boldsymbol{U} = \boldsymbol{U}_a \, \boldsymbol{U}_m$ and $\hat{\boldsymbol{U}} = \boldsymbol{U}_a \, \hat{\boldsymbol{U}}_m$, no actual calculations, just an expansion introducing zeros at the determinate constraints degrees of freedom;
2. $\boldsymbol{P}_{iu} = -(\boldsymbol{M}_{fe} \, \hat{\boldsymbol{U}} - \boldsymbol{M}_{md} \, (\boldsymbol{M}_{md}^T \hat{\boldsymbol{U}}) \; + \boldsymbol{C}_{fe}\boldsymbol{U}$, exactly as given;
3. $\boldsymbol{P}_{iu} = \boldsymbol{U}_a^T \; \boldsymbol{P}_{iu}$, no actual calculations, just a compression by removing zeros at the determinate constraints degrees of freedom;
4. solve $\boldsymbol{K}_u \, \boldsymbol{U}_m = \boldsymbol{P}_{iu}$, with a forward-backward substitution using $\boldsymbol{L}_k$.
5. overwrite: $\hat{\boldsymbol{U}} = \boldsymbol{U}$.

Once all the desired mean axes modes $\boldsymbol{U}_m$ are available their absolute nodal equivalents are obtained at once with: $\boldsymbol{U} = \boldsymbol{T} \, \boldsymbol{U}_a \, \boldsymbol{U}_m$, using the same sparsity preserving calculations previously seen.

A few application examples

1. We will now verify the eigensolutions in mean axes with a trivial, hopefully enlightening, example, related to three masses, m, connected by two springs, k, free to vibrate along a straight line without any damping-friction, for which we can write:
$$\boldsymbol{M}_{fe} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \; \boldsymbol{K}_{fe} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \; \boldsymbol{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$
The related eigensolutions can easily verified by hand as being:
$$\omega_r^2 = 0 \quad \omega_{d1}^2 = \frac{k}{m} \quad \omega_{d2}^2 = 3\frac{k}{m} \quad \boldsymbol{u}_r = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \quad \boldsymbol{U}_d = \begin{bmatrix} -1 & -1 \\ 0 & 2 \\ 1 & -1 \end{bmatrix}$$
We then define $\boldsymbol{R}$ and $\boldsymbol{U}_a$, Eq. (39), along with the calculation of the core attached to mean axes transformation matrix $\boldsymbol{T}$, Eq. (49):
$$\boldsymbol{R} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \; \boldsymbol{U}_a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \; \boldsymbol{T} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$
Consequently the resulting matrices of the deformable part in mean axes, Eq. (55), are:
$$\boldsymbol{M}_{mu} = \frac{m}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \; \boldsymbol{K}_u = k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$
From what above the single deformable mode is given below, along with its back transformation to physical nodal displacements:
$$\omega_{d1}^2 = \frac{k}{m} \quad \omega_{d2}^2 = 3\frac{k}{m} \quad \boldsymbol{U}_m = \begin{bmatrix} 1/3 & 1 \\ 2/3 & 0 \end{bmatrix} \quad \boldsymbol{U}_d = \boldsymbol{T}\boldsymbol{U}_a\boldsymbol{U}_m = \begin{bmatrix} -1 & -1 \\ 0 & 2 \\ 1 & -1 \end{bmatrix}$$
It can be seen we have obtained what we should.

2. After having seen that within a reasonably fine FE model it would be ineffective to calculate $\boldsymbol{F}_a$ and $\boldsymbol{M}_u$ explicitly to proceed to the symbolically simpler direct use of the power iteration of Eq. (63) used as:

$$\boldsymbol{U}_m^{k+1} = \boldsymbol{F}_a \ \boldsymbol{M}_u \ \boldsymbol{U}_m^k \tag{66}$$

we contradict ourselves by discovering that it could not be so always. That happens when $\boldsymbol{F}_a$ is available directly from experimental measurement, stiffness matrices are never (rarely) measured. In such a case, assuming a precisely known mass distribution is available and the influence coefficients of $\boldsymbol{F}_a$ are measured at enough points to make it acceptable a lumping of the masses, to produce a mass matrix equivalent to $\boldsymbol{M}_{fe}$, since $\boldsymbol{R}$ is precisely available by inspection, it is possible to calculate $\boldsymbol{M}_u$ so that, starting from relatively simple static only measurements, Eq. (66) leads to a quasi experimental determination of free deformable vibration modes. It might appear as a kind of vintage approach to modal testing but, by avoiding the difficulties of approximating free-free conditions, still remains quite a valid alternative to a complete free-free dynamic test, especially in the case of scaled dynamic models.

3. A similar approach remains useful even when it is convenient to carry out dynamic vibration tests on a statically determinate structure and it is wanted to recover free-free modes afterward [19]. We assume, once more, that the mass distribution is known very well, the constrained modes shapes, called $\boldsymbol{V}$, are measured at enough points to allow calculating $\boldsymbol{M}_u$ with an adequate precision after lumping the known mass distribution, into an equivalent $\boldsymbol{M}_{fe}$ again. So the constraint dynamic modal test will provide a set of ω_i^2, $\boldsymbol{V}$, and possibly the generalized modal masses $\text{Diag}\{m_i\}$. If the latter are not known then they can be computed using $\text{Diag}\{m_i\} = \boldsymbol{V}^T \ \boldsymbol{M}_{fe} \ \boldsymbol{V}$, an operation likely to be carried out anyhow to cross check the orthogonality of the measured modes. Retaining the constraint modal amplitudes as being $\boldsymbol{u}_m$, all the needed ingredients can be computed with the following:

$$\boldsymbol{F}_a = \text{Diag}\left\{\frac{1}{m_i \omega_i^2}\right\} \tag{67}$$

$$\boldsymbol{M}_u = \text{Diag}\{m_i\} - \boldsymbol{V}^T \ \boldsymbol{M}_{fe} \ \boldsymbol{R} \ \boldsymbol{M}_r^{-1} \ \boldsymbol{R}^T \ \boldsymbol{M}_{fe} \ \boldsymbol{V} \tag{68}$$

so that Eq. (66) can be used much as before. The mode shapes at the measurement points are then given by: $\boldsymbol{u} = \boldsymbol{V} \ \boldsymbol{u}_m$. To reduce errors and improve convergence the described procedure requires to measure a bit more constrained frequencies and modes than the desired free-free ones.

5. Concluding remarks

The problem of setting up an eigensolution path separating the calculation of the undamped deformable vibration modes of a free-free structures from rigid body modes, a need extensively explained in the introduction, is embedded in the use of mean axes. In fact the needed separation is nothing but their definition. Despite being it straightforward coming to such a conclusion it is not as simple to have the related matter easily mastered by students. Often they feel a circulatory and confusing interplay of the very definition of mean axes with free-free vibration modes, which embed mean axes by nature, to the point of thinking that the two things are the same. Trying to settle the matter by pointing them to the literature clashes with largely scattered clues, requiring too much time to be synthesized in a unified and simple framework to be mastered not only for deformable vibrations modes. The need of writing this tutorial is mainly due to the afore mentioned difficulty in gathering the whole content of what here presented through too many streams, scattered and/or delivering to different rivers, flowing in the literature. Consequently, starting from the excuse of computing deformable modes through mean axes, contained in the introduction, this tutorial is much dedicated to developing them, so adhering to its title. The more general attitude taken to tackle the problems should not miss the objective related to the calculation of deformable modes but makes it just a single facet of a general framework unifying a larger set of dynamic problems, e.g: calculation of frozen snapshots of dynamic trim conditions and related stress recovery, for a free structure. In such a way it should have provided the reader with a well furnished toolbox from which the best wrench can be chosen for any related pertaining problem at hand. Some past experience in using the content of this tutorial in structural dynamics and aeroelasticity supports the taken path. In particular, the repetition of the concept by presenting it through an RA followed by a specialization to FE should meets two objectives. At first an RA allows a mastering of the subject through simple assignments, workable by hand, or with little usage of a computer and numerical skills, so to grasp a direct physical feeling of what is being done. The following extension to FE, which is the way the related concepts are used in real calculations nowadays, with much computer and numerical sophistication, makes it clear that what has been mastered though simpler application is viable for real applications too. Without coding yet another tutorial FE code, the possibility for a student to exercise her/his numerical skill with assignment using FE models, is quite easy for not so large models and with FE codes allowing to export the related nodal matrices and rigid body modes into a problem solving environment furnished with sparse matrix tools, e.g. ScicosLab and Octave. Using them it is possible to repeat the steps presented in the simple example shown above, with all the here explained sparse numerical care, checking the results so obtained with the full modal analyses of the parent FE code. Much the same can be done for dynamic trim snapshots and modes acceleration. Of significant help is combining a fully blown RA application, of a relatively simple structure, with a related fine FE model, so to verify own capabilities in setting up simple approximations apt to well represent the physics of an assigned problem. There might be some added value also, but the support for such a claim is left to the reader.

References

[1] H. Baher. *Analog and digital signal processing.* Wiley, New York, 2002.

[2] R.L. Bisplinghoff, H. Ashley, and R.L. Halfman. *Aeroelasticity.* Dover, New York, 1996.

[3] M. Borri and P. Mantegazza. Efficient solution of quadratic eigenproblems arising in dynamic analysis of structures. *Computer Methods in Applied Mechanics and Engineering*, 12:19–31, 1977.

[4] Y.T. Chung and L.L. Kahrch. A general procedure for finite element model check and model identification. *MSC report*, 1986.

[5] H. L. Cox. Vibration of missiles: Matric formulation of the problem involving free and harmonically forced flexural vibrations. *Aircraft Engineering*, 33(383):2–7, january 1961. part 1.

[6] H. L. Cox. Vibration of missiles: Matric formulation of the problem involving free and harmonically forced flexural vibrations. *Aircraft Engineering*, 33(384):48–55, february 1961. part 2.

[7] R. R. Craig and M. C. C. Bampton. Coupling of substructures for dynamic analysis. *AIAA Journal*, 6(7):1313–1319, 1968.

[8] B. M. Fraeijs de Veubeke. Iteration in semi definite eigenvalue problems. *Journal of the Aeronautical Sciences*, 22(10):700–720, 1955.

[9] B. M. Fraeijs de Veubeke. Nonlinear dynamics of flexible bodies. In *Dynamic and Control of Non-rigid Spacecraft*, 1976.

[10] B. M. Fraeijs de Veubeke, M. Geradin, A. Huckand, and M.A. Hogge. *Structural dynamics - Heat Conduction*. International Centre for Mechanical Sciences, Springer and Verlag, 1972.

[11] A.R. Dusto, G.W. Brune, G.M. Dornfeld, J.E. Mercer, S.C. Pilet, P.E. Rubbert, R.C. Schwanz, P. Smutny, E.N. Tinoco, and J.A. Weber. A method for predicting the stability derivatives of an elastic aircraft, static aeroelastic characteristics. Technical Report NASA CR-114712, NASA, 1974. Vol. I - Flextab Theoretical description.

[12] M. Geradin and D. Rixen. *Mechanical vibrations: theory and application to structural dynamics*. Wiley, 1997.

[13] G.H. Golub and C.F. Van Loan. *Matrix Computations*. Johns Hopkins University Press, Baltimore, London, 1996.

[14] A. Jennings. Natural vibrations of free structures. *Aircraft Engineering*, 34(397):81–83, march 1962.

[15] H. Lamb. *Higher Mechanics*. Cambridge University Press, 1929.

[16] L. Meirovitch and I. Tuczu. The lure of mean axes. *ASME journal*, 74:497–504, May 2007.

[17] R. D. Milne. Dynamics of the deformable airplane. Technical report, Ministry of Aviation, Aeronautical Research Council Reports and Memoranda, 1964.

[18] R.L. Taylor O.C. Zienkiewicz. *Theory of Matrix Structural Analysis*. Elsevier Butterworth Heinemann, 2005.

[19] J.S. Przemieniecki. *Theory of Matrix Structural Analysis*. Dover, New York, 1985.

[20] J.N. Reddy. *Applied Functional Analysis and Variational Methods in Engineering*. McGraw-Hill and Krieger (Melbourne), 1986 and 1991.

[21] W.P. Rodden and J.R. Love. Equations of motion of a quasi steady flight vehicle utilizing restrained static aeroelastic characteristics. *Journal of Aircraft*, 22:802–809, 1985.

[22] Y. Saad. *Numerical Methods for Large Eigenvalue Problems*. http://www-users.cs.umn.edu/˜saad/books.html, 2011.

[23] A.E. Stockwell. A verification procedure for msc/nastran finite element models. *NASA-CR-4675*, 1995.

[24] F. Tilman, R. Galletly, and J. Zins. Msc/nastran model checkout. In *MSC/NASTRAN User's Conference*, 1986.

[25] J. H. Wilkinson. *The algebraic eigenvalue problem*. Oxford University Press, USA, 1965.

[26] J. H. Wilkinson. The solution of ill-conditioned linear equations. In H.S. Wilf A. Ralston, editor, *Mathematical Methods for Digital Computers*, volume 2. Wiley, 1967.

[27] D. Williams. *An introduction to the theory of aircraft structures*. Edward Arnold Ltd., London, 1960.