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TESTING IDENTIFYING ASSUMPTIONS IN FUZZY REGRESSION DISCONTINUITY DESIGNS*

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ABSTRACT. We propose a new specification test for assessing the validity of fuzzy regression discontinuity designs (FRD-validity). We derive a new set of testable implications, characterized by a set of inequality restrictions on the joint distribution of observed outcomes and treatment status at the cut-off. We show that this new characterization exploits all of the information in the data that is useful for detecting violations of FRD-validity. Our approach differs from and complements existing approaches that test continuity of the distributions of running variables and baseline covariates at the cut-off in that we focus on the distribution of the observed outcome and treatment status. We show that the proposed test has appealing statistical properties. It controls size in a large sample setting uniformly over a large class of data generating processes, is consistent against all fixed alternatives, and has non-trivial power against some local alternatives. We apply our test to evaluate the validity of two FRD designs. The test does not reject FRD-validity in the class size design studied by Angrist and Lavy (1999) but rejects it in the insurance subsidy design for poor households in Colombia studied by Miller, Pinto, and Vera-Hernández (2013) for some outcome variables. Existing density continuity tests suggest the opposite in each of the two cases.

Keywords: Fuzzy regression discontinuity design, nonparametric test, inequality restriction, multiplier bootstrap.

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$*$. This paper merges and replaces the unpublished independent works of Arai and Kitagawa (2016) and Hsu, Mourifié, and Wan (2016).

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1. INTRODUCTION

Regression discontinuity (RD) design, first introduced by Thistlethwaite and Campbell (1960), is one of the most widely used quasi-experimental methods in program evaluation studies. The RD design exploits discontinuity in treatment assignment due to administrative or legislative rules based on a known cut-off of an underlying assignment variable, which we refer to as the running variable. The RD design is called sharp if the probability of being treated jumps from zero to one, and is called fuzzy otherwise. See Imbens and Lemieux (2008), and Lee and Lemieux (2010) for reviews, and Cattaneo and Escanciano (2017) for recent advances of the literature.

The RD design identifies the causal impact of the treatment by comparing the outcomes of treated and non-treated individuals close to the cut-off. The validity of the RD design relies crucially on the assumption that those individuals immediately below the cut-off have the same distribution of unobservables as those individuals immediately above the cut-off. The first formalization of this argument appears in Hahn, Todd, and Van der Klaauw (2001, HTV hereafter), which utilises a potential outcomes framework to establish identification of causal effects at the cut-off. Subsequently, Frandsen, Frölich, and Melly (2012, FFM hereafter), Dong and Lewbel (2015), and Cattaneo, Keele, Titiunik, and Vazquez-Bare (2016) consider a refined set of identifying conditions. In the fuzzy regression discontinuity design (FRD) setting, the two key conditions for identification, which we refer to as *FRD-validity*, are (i) *local continuity*, the continuity of the distributions of the potential outcomes and treatment selection heterogeneity at the cut-off, and (ii) *local monotonicity*, the monotonicity of the treatment selection response to the running variable at the cut-off.

The credibility of FRD-validity is controversial in many empirical contexts. For instance, agents (or administrative staff) may manipulate the value of the running variable to be eligible for their preferred treatment. If their manipulation depends on their underlying potential outcomes, this can lead to a violation of the local continuity condition. Even when manipulation of the running variable is infeasible or absent, the local continuity condition can fail if the distribution of unobservables is discontinuous across the cut-off. This is a common concern in empirical research, for instance, when the RD design exploits geographical boundaries across which individuals are less likely to relocate but the ethnic distribution of the population is changing discontinuously. See Dell (2010), and Eugster, Lalive, Steinhauer, and Zweimüller (2017). As a related example, violation of local continuity becomes a concern when multiple programs share an index of treatment assignment and its threshold (e.g. the poverty line, state borders, etc.), but an individual's treatment status is observed only for the treatment of interest. See Miller, Pinto, and Vera-Hernández (2013), Carneiro and Ginja (2014), and Keele and Titiunik (2015) for examples and discussions of this issue.

Motivated by a clearer economic interpretation and the availability of testable implications, Lee (2008) imposes a *stronger* set of identifying assumptions that implies continuity of the distributions of the running variable and covariates at the cut-off. Following his approach, researchers routinely assess the continuity condition by applying the tests of McCrary (2008), Otsu, Xu, and Matsushita (2013), Cattaneo, Jansson, and Ma (2020), and Canay and Kamat (2018). When the running variable is manipulated, Gerard, Rokkanen, and Rothe (2020) provides a partial identification approach in the presence of "one-sided manipulation." As noted by McCrary (2008), however, in the absence of Lee's additional identifying assumption, the continuity of the distributions of the running variable and baseline covariates at the cut-off is neither necessary nor sufficient for FRD-validity, and rejection or acceptance of the existing tests is not informative about FRD-validity or violation thereof.

This paper proposes a novel test for FRD-validity. We first derive a new set of testable implications, characterized by a set of inequality restrictions on the joint distribution of observed outcomes and treatment status at the cut-off. These testable implications are necessary conditions for FRD-validity, while we show they are sharp in the sense that they cannot be strengthened without additional assumptions. We propose a nonparametric test for these testable implications. The test controls size uniformly over a large class of distributions of observables, is consistent against all fixed alternatives violating the testable implications, and has non-trivial power against some local alternatives. Implementability and asymptotic validity of our test neither restricts the support of Y nor presumes continuity of the running variable's density at the cut-off.

The testable implication that our test assesses differs from and complements the testable implication that the existing density continuity tests focus on. As we illustrate in Section 2.3, there are important empirical contexts where the results of the existing approach are not informative about FRD-validity while ours are. They include scenarios where the distribution of unobservables is discontinuous at the cut-off, multiple programs share the same running variable and the same threshold, and manipulation of the running variable is driven by factors independent of the potential outcomes. It is also important to note that our testable implication assesses local monotonicity, about which the existing approach is not informative. The novelty of our approach is that it exploits those aspects of the data that are informative in assessing FRD-validity but that have been neglected by the existing density continuity approach. We therefore recommend that our test is implemented alongside existing tests for continuity of the running variable density, regardless of the results thereof.

To illustrate our proposal, we apply our test to the designs studied in Angrist and Lavy (1999), and Miller, Pinto, and Vera-Hernández (2013). Angrist and Lavy (1999) use the discontinuity of class size with respect to enrollment due to Maimonides' rule to identify the causal effect of class size on student performance. We do not find statistically significant violation of our new testable implication for FRD-validity for any of the four outcome variables (Grade 4 Math and Verb, Grade 5 Math and Verb). In contrast, the existing continuity test suggests statistically significant evidence for discontinuity of the running variable's density at the cut-off (see Otsu, Xu, and Matsushita, 2013). Miller, Pinto, and Vera-Hernández (2013) evaluate the impact of "Colombia's Régimen Subsidiado"– a publicly financed insurance program–on 33 outcomes, where program eligibility is determined by a poverty index. Since our approach makes use of observations of not only the running variable but also of treatment status and the observed outcome, it has the unique feature of being outcome-specific, i.e. when multiple outcomes are studied within the same FRD design, researchers can assess credibility of FRD-validity separately for each outcome variable. In this example, the continuity test supports continuity of the running variable density at the cut-off, while we find statistically significant evidence for the violation of our new testable implication for FRD-validity for 3 outcome variables (Household Education Spending, Total Spending on Food, and Total Monthly Spending). This result suggests further investigation would be beneficial for identifying and estimating the causal effect on these outcomes.

The rest of the paper is organized as follows. In Section 2, we lay out the main identifying assumptions that our test aims to assess and derive their testable implications. Section 3 provides test statistics and shows how to obtain their critical values. Monte Carlo experiments shown in Section 4 examine the finite sample performance of our tests. Section 5 presents the empirical applications. Section 6 concludes the paper. The Supplemental Material (Arai, Hsu, Kitagawa, Mourifié, and Wan (2020)) provides detailed discussion of how our test differs and complements existing tests, several extensions, the asymptotic validity of our test, all proofs, and additional empirical results.

2. Identifying Assumptions and Sharp Testable Implications

2.1. Setup and Notation.

We adopt the potential outcome framework introduced in Rubin (1974). Let $(\Omega, \mathcal{F}, P)$ be a probability space, where we interpret Ω as the population of interest and $\omega \in \Omega$ as a generic individual in the population.

Let R be an observed continuous random variable with support $\mathcal{R} \subset \mathbb{R}$.[1] We call R the *running variable*. Let $D(\cdot,\cdot) : \mathcal{R} \times \Omega \to \{0,1\}$ and $D(r,\omega)$ be the *potential treatment* that individual ω would have received, had her running variable been set to r. For $d \in \{0,1\}$, we define mappings $Y_d(\cdot,\cdot) : \mathcal{R} \times \Omega \to \mathcal{Y} \subset \mathbb{R}$ and let $Y_d(r,\omega)$ denote the *potential outcome* of individual ω had her treatment and running variable been set to d and r, respectively.

We view $(Y_1(r,\cdot), Y_0(r,\cdot), D(r,\cdot))_{r\in\mathcal{R}}$ as random elements indexed by r and write them as $(Y_1(r), Y_0(r), D(r))$ when it causes no confusion. By definition, $D(R) \in \{0,1\}$ is the observed treatment and we abbreviate it as D. Likewise, we denote the observed outcome by $Y = Y_1(R)D(R) + Y_0(R)(1 - D(R))$ throughout the paper. We use P to denote the joint distribution of $\big((Y_1(r), Y_0(r), D(r))_{r\in\mathcal{R}}, R\big)$, which induces the joint distribution of observables (Y, D, R).[2] We assume throughout that the conditional distribution of (Y, D) given $R = r$ is well-defined for all r in some neighborhood of r_0, and that $\lim_{r\downarrow r_0} D(r)$ and $\lim_{r\uparrow r_0} D(r)$ are well defined for all ω. Note that by letting the potential outcomes be indexed by r, we allow the running variable to have a direct causal effect on outcomes. This could be relevant in some empirical applications as discussed in Dong and Lewbel (2015), and Dong (2018).

Analogous to the local average treatment effect (LATE) framework (Imbens and Angrist (1994)), we define the compliance status $T(r,\omega)$ of individual ω in a small neighborhood of the cut-off r_0 based on how the potential treatment varies with r. Similar to FFM, Bertanha and Imbens (2020), and Dong and Lewbel (2015), for $\epsilon > 0$, we classify the population members into one of the following

[1] In this paper we consider a continuous running variable. Kolesár and Rothe (2018) study inference on ATE in the sharp regression discontinuity designs with a discrete running variable.

[2] For the purpose of exposition, we do not introduce other observable covariates X here. Appendix C.2 of the Supplemental Material incorporates X into the analysis.

five categories:

$$T_\epsilon(\omega) = \begin{cases} \mathbf{A}, & \text{if } D(r,\omega) = 1, \text{ for } r \in (r_0 - \epsilon, r_0 + \epsilon), \\ \mathbf{C}, & \text{if } D(r,\omega) = 1\{r \geq r_0\}, \text{ for } r \in (r_0 - \epsilon, r_0 + \epsilon), \\ \mathbf{N}, & \text{if } D(r,\omega) = 0, \text{ for all } r \in (r_0 - \epsilon, r_0 + \epsilon), \\ \mathbf{DF}, & \text{if } D(r,\omega) = 1\{r < r_0\}, \text{ for } r \in (r_0 - \epsilon, r_0 + \epsilon), \\ \mathbf{I}, & \text{otherwise} \end{cases}, \tag{1}$$

where **A, C, N, DF** and **I** represent "always takers", "compliers", "never takers", "defiers" and "indefinite", respectively.[3]

2.2. **Identifying Assumptions and Testable Implication.** We present the main identifying assumptions and their testable implications. In the statement of the assumptions we assume that all the limiting objects exist.

Assumption 1 (Local monotonicity). *For $t \in \{\mathbf{DF}, \mathbf{I}\}$, $\lim_{\epsilon \to 0} P(T_\epsilon = t | R = r_0 + \epsilon) = 0$ and $\lim_{\epsilon \to 0} P(T_\epsilon = t | R = r_0 - \epsilon) = 0$.*

Assumption 2 (Local continuity). *For $d = 0, 1$, $t \in \{\mathbf{A}, \mathbf{C}, \mathbf{N}\}$, and any measurable subset $B \subseteq \mathcal{Y}$,*

$$\lim_{\epsilon \to 0} P(Y_d(r_0 + \epsilon) \in B, T_\epsilon = t | R = r_0 + \epsilon) = \lim_{\epsilon \to 0} P(Y_d(r_0 - \epsilon) \in B, T_\epsilon = t | R = r_0 - \epsilon).$$

Assumptions 1 and 2 play similar roles to the instrument monotonicity and instrument exogeneity (exclusion and random assignment) assumptions in the LATE framework. Assumption 1 says that as the neighborhood of r_0 shrinks, the conditional proportion of defiers and indefinites converges to zero, implying that only "always takers", "compliers", and "never takers" may exist at the limit. The local continuity assumption says that the conditional joint distributions of potential outcomes and compliance types are continuous at the cut-off. Our local continuity condition concerns distributional continuity rather than only continuity of the conditional mean (and so is unlike HTV).

The main feature of FRD designs is that the probability of receiving treatment is discontinuous at the cut-off. To be consistent with the local monotonicity assumption, we specify the discontinuity so that the propensity score jumps *up* as r goes above the cut-off.

[3] The above definition coincides with the definition of types in FFM as $\epsilon \to 0$. As pointed out by Dong and Lewbel (2015), for a given ϵ and a given individual ω, this definition implicitly assumes the group to which ω belongs does not vary with r. This way of defining the treatment selection heterogeneity does not restrict the shape of $P(D = 1 | R = r)$ over $(r_0 - \epsilon, r_0 + \epsilon)$.

Assumption 3 (Discontinuity)**.** $\pi^+ \equiv \lim_{r \downarrow r_0} P(D = 1|R = r) > \lim_{r \uparrow r_0} P(D = 1|R = r) \equiv \pi^-$.

Under Assumptions 1 to 3, the compliers' potential outcome distributions at the cut-off, defined as

$$
\begin{aligned}
F_{Y_1(r_0)|\mathbf{C},R=r_0}(y) &\equiv \lim_{r \to r_0} P\left(Y_1(r) \le y | T_{|r-r_0|} = \mathbf{C}, R = r\right), \\
F_{Y_0(r_0)|\mathbf{C},R=r_0}(y) &\equiv \lim_{r \to r_0} P\left(Y_0(r) \le y | T_{|r-r_0|} = \mathbf{C}, R = r\right),
\end{aligned}
$$

are identified by the following quantities:[4] for all $y \in \mathcal{Y}$,

$$
\begin{aligned}
F_{Y_1(r_0)|\mathbf{C},R=r_0}(y) &= \frac{\lim_{r \downarrow r_0} \mathbb{E}_P[1\{Y \le y\}D|R = r] - \lim_{r \uparrow r_0} \mathbb{E}_P[1\{Y \le y\}D|R = r]}{\pi^+ - \pi^-}, \\
F_{Y_0(r_0)|\mathbf{C},R=r_0}(y) &= \frac{\lim_{r \uparrow r_0} \mathbb{E}_P[1\{Y \le y\}(1-D)|R = r] - \lim_{r \downarrow r_0} \mathbb{E}_P[1\{Y \le y\}(1-D)R = r]}{\pi^+ - \pi^-}.
\end{aligned}
$$

This is analogous to the distributional identification result by Imbens and Rubin (1997) for the LATE model. The identification of the compliers' potential outcome distributions implies the identification of a wide class of causal parameters including the average effect amongst the compliers and local quantile treatment effects.[5] Our identification result modifies FFM's Lemma 1 to accommodate the fact that we do not exclude r from the potential outcomes.

Note that Assumption 3 can be tested using the inference methods proposed by Calonico, Cattaneo, and Titiunik (2014), and Canay and Kamat (2018). We therefore focus on testing Assumptions 1 and 2.

The next theorem shows that *local monotonicity* and *local continuity* together imply a set of inequality restrictions on the distribution of data.

Theorem 1. *(i) Under Assumptions 1 and 2, the following inequalities hold:*

$$\lim_{r \uparrow r_0} \mathbb{E}_P[1\{y \le Y \le y'\}D|R = r] - \lim_{r \downarrow r_0} \mathbb{E}_P[1\{y \le Y \le y'\}D|R = r] \le 0 \tag{2}$$

$$\lim_{r \downarrow r_0} \mathbb{E}_P[1\{y \le Y \le y'\}(1-D)|R = r] - \lim_{r \uparrow r_0} \mathbb{E}_P[1\{y \le Y \le y'\}(1-D)|R = r] \le 0 \tag{3}$$

[4]For completeness, we show this identification result in Proposition E.1 in Appendix E.2 of the Supplemental Material.

[5]Assumptions 1 and 2 play similar roles to FFM's Assumptions I3 and I2, respectively. The main difference from FFM's assumptions is that FFM define the compliance status solely at the limit, and assume that the conditional distributions of the potential outcomes given the limiting compliance status and the running variable are continuous at the cut-off.

for all $y, y' \in \mathbb{R}$.

(ii) For a given distribution of observables (Y, D, R), assume that the conditional distribution of Y given (D, R) has a probability density function with respect to a dominating measure μ on $\mathcal{Y}$, has an integrable envelope with respect to μ, and whose left-limit and right-limit with respect to the conditioning variable R are well defined at $R = r_0$, $\mu-$a.s. If inequalities (2) and (3) hold, there exists a joint distribution of $(\tilde{D}(r), \tilde{Y}_1(r), \tilde{Y}_0(r) : r \in \mathcal{R})$ such that Assumptions 1 and 2 hold, and the conditional distribution of $\tilde{Y} = \tilde{Y}_1(R)\tilde{D}(R) + \tilde{Y}_0(R)(1 - \tilde{D}(R))$ and $\tilde{D} = \tilde{D}(R)$ given $R = r$ induces the conditional distribution of (Y, D) given $R = r$ for all $r \in \mathcal{R}$.

Theorem 1 (i) shows a necessary condition that the distribution of observable variables has to satisfy under the FRD-validity conditions. In other words, a violation of inequalities (2) and (3) is informative that at least one of the FRD-validity conditions is violated. Theorem 1 (ii) clarifies that inequalities (2) and (3) are the most informative way to detect all of the observable violations of the FRD-validity assumptions and the testable implications cannot be strengthened without making further assumptions. We emphasize, however, that FRD-validity is a refutable but not a *confirmable* assumption, i.e., finding inequalities (2) and (3) hold in data does not guarantee FRD-validity.

Similar to the testable implications of the LATE model considered in Balke and Pearl (1997), Imbens and Rubin (1997), Heckman and Vytlacil (2005), Kitagawa (2015), and Mourifié and Wan (2017), the testable implications of Theorem 1 (i) can be interpreted as an FRD version of *non-negativity of the potential outcome density functions for the compliers at the cut-off.* Despite such an analogy, the framework and features specific to RD designs give rise to some important differences and challenges. First, the assumption that we test is continuity of the conditional distributions of the potential outcomes and compliance status local to the cut-off, rather than the global exclusion or no-defier restrictions of the standard LATE model. Second, since the testable implications concern distributional inequalities local to the cut-off, the construction of the test statistic requires proper smoothing with respect to the conditioning running variable.

2.3. **How does our testable implication differ from existing implications?** FRD-validity, as defined by Assumptions 1 and 2, does not constrain the marginal density of R to be continuous at the cut-off. This contrasts with the testable implications of continuity of the running variable and covariate densities obtained in Lee (2008), and Dong (2018), which hinge on a stronger restriction such that the density of the running variable given the potential outcomes is continuous at the cut-off. See McCrary (2008), Otsu, Xu, and Matsushita (2013), Cattaneo, Jansson, and Ma (2020), and Bugni and Canay (2018) for tests of the continuity of the running variable density, and Canay and Kamat (2018) for tests of the continuity of the covariate densities.

The testable implication of Theorem 1 (i) is valid no matter whether one assumes such an additional restriction or not. The testable implication concerns the joint distribution of (Y, D) local to the cut-off, which the existing approach of assessing continuity of the densities of the running variable and observable covariates does not make use of. In this sense, our approach, which does not require continuity of the running variable's density, complements the existing approach of using continuity tests and we recommend the implementation of our test (proposed below) in any FRD studies, whatever results the existing continuity tests yield.

There are several important empirical contexts where supporting or rejecting continuity of the running variable's density is not informative about FRD-validity, while the testable implication of Theorem 1 (i) can be. First, even when the running variable's density is known to be continuous, it is still often controversial to assume that the distribution of unobservable heterogeneity affecting the outcomes is continuous at the cut-off. For instance, when an RD design exploits geographical or language boundaries (e.g., Dell (2010), and Eugster, Lalive, Steinhauer, and Zweimüller (2017)), the distribution of (unobservable) ethnicity may change discontinuously, even though individuals are distributed smoothly over the space. If the discontinuity of the distribution of unobservables leads to violation of the testable implication of Theorem 1 (i), our approach correctly refutes FRD-validity.

Second, if multiple programs share the same running variable and the same threshold (compound treatments), an FRD design that ignores the other programs can lead to violation of continuity of the potential outcome distributions (for the program of interest), even when the density of the running variable is continuous. For instance, empirical scenarios that rely on a spatial regression discontinuity design exploiting jurisdictional, electoral, or market boundaries (see Keele and Titiunik (2015), and references therein) can violate local continuity in this way. The issue of compound treatments is also of concern when multiple social programs targeted at the poor assign their eligibility according to a common poverty index and poverty line. (Carneiro and Ginja (2014)).

Third, in contrast to the previous two contexts, discontinuity of the running variable's density does not necessarily imply violation of local continuity if manipulation of the running variable is independent of the potential outcomes (possibly conditional on observable covariates). In this case, the testable implication of Theorem 1 (i) does not refute FRD-validity even though the running variable's density is discontinuous. See our empirical application in Section 5.1, below. In addition, Appendix B of the Supplemental Material provides detailed analytical comparisons between the testable implications of Lee (2008) and ours.

Another distinguishing feature of our approach is that our testable implication can also detect violation of local monotonicity. It is therefore valuable to assess the testable implication also in those scenarios where local continuity is credible while local monotonicity is less credible. Examples include studies examining the returns to field of study or college major, exploiting discontinuity generated by a centralized score-based admission system (Hastings, Neilson, and Zimmerman (2014), and Kirkeboen, Leuven, and Mogstad (2016)). In this context, the validity of local monotonicity can be a concern if an individual's choice of treatment (e.g., graduating with a degree in science rather than a degree in humanities) is different from their initial assignment in the program (e.g., admitted to a science or humanities program). Defiers can exist if some students always switch from their assigned major based on revisions of their beliefs or preferences.[6]

3. Testing Procedure

This section proposes a testing procedure for the testable implications of Theorem 1 (i). We assume that a sample consists of independent and identically distributed (i.i.d.) observations, $\{(Y_i, D_i, R_i)\}_{i=1}^n$. Noting that the inequality restrictions of Theorem 1 (i) amount to an infinite number of unconditional moment inequalities local to the cut-off, we adopt and extend the inference procedure for conditional moment inequalities developed in Andrews and Shi (2013) by incorporating the local feature of the RD design.[7] The implementation and asymptotic validity of our test neither restricts the support of Y nor presumes continuity of the running variable's density at the cut-off. See Appendix D of the Supplemental Material for regularity conditions and the asymptotic validity of our test.

[6]See Zafar (2011), and Stinebrickner and Stinebrickner (2014) for empirical evidence on how college students form and revise their beliefs on own academic outcomes for their majored and non-majored subjects and how this relates to their subsequent switch of majors.

[7]Other approaches and recent advances of the inference of conditional moment inequalities include Chernozhukov, Lee, and Rosen (2013), Armstrong and Chan (2016), and Chetverikov (2018). The methods proposed in these works are free from the infinitesimal uniformity factor η in Algorithm 1. Formal investigation of their applicability to the current regression discontinuity context is beyond the scope of this paper.

Consider a class of instrument functions $\mathcal{G}$ indexed by $\ell \in \mathcal{L}$:

$$\mathcal{G} = \left\{ g_\ell(\cdot) = 1\{\cdot \in C_\ell\} : \ell \equiv (y, y') \in \mathcal{L} \right\}, \text{ where}$$
$$C_\ell = [y, y'] \cap \mathcal{Y},$$
$$\mathcal{L} = \{(y, y') : -\infty \leq y \leq y' \leq \infty\}.$$

$\mathcal{G}$ consists of indicator functions of closed and connected intervals on $\mathcal{Y}$. Expressing the inequalities (2) and (3) by

$$\begin{aligned} \nu_{P,1}(\ell) &\equiv \lim_{r\uparrow r_0} \mathbb{E}_P[g_\ell(Y)D|R=r] - \lim_{r\downarrow r_0} \mathbb{E}_P[g_\ell(Y)D|R=r] \leq 0, \\ \nu_{P,0}(\ell) &\equiv \lim_{r\downarrow r_0} \mathbb{E}_P[g_\ell(Y)(1-D)|R=r] - \lim_{r\uparrow r_0} \mathbb{E}_P[g_\ell(Y)(1-D)|R=r] \leq 0, \end{aligned} \tag{4}$$

for all $\ell \in \mathcal{L}$, we set up the null and alternative hypotheses as

$$\begin{aligned} &H_0: \ \nu_{P,1}(\ell) \leq 0 \text{ and } \nu_{P,0}(\ell) \leq 0 \text{ for all } \ell \in \mathcal{L}, \\ &H_1: \ H_0 \text{ does not hold.} \end{aligned} \tag{5}$$

Noting that H_0 is equivalent to $\sup_{d\in\{0,1\},\ell\in\mathcal{L}} \omega_d(\ell)\nu_{P,d}(\ell) \leq 0$ for a positive weight function $\omega_d(\ell) > 0$, we construct our test statistic by plugging in estimators of $\nu_{P,d}(\ell)$ weighted by the inverse of its standard error estimate.

We construct $\hat{\nu}_d(\ell)$, an estimator for $\nu_{P,d}(\ell)$, as the difference of the two local linear regressions estimated from below and above the cut-off. We do not vary the bandwidths over $\ell \in \mathcal{L}$, but we allow them to vary across the cut-offs; let $h_+ = c_+ h$ and $h_- = c_- h$ be the bandwidths above and below the cut-off, respectively. We assume that their convergence rates with respect to the sample size n are common, as specified by h, e.g., $h = n^{-1/4.5}$. The difference between h_+ and h_- can be captured by possibly distinct constants c_+ and c_-.

Let $\sigma_{P,d}(\ell)$ be the asymptotic standard deviation of $\sqrt{nh}(\hat{\nu}_d(\ell) - \nu_{P,d}(\ell))$ and $\hat{\sigma}_d(\ell)$ be a uniformly consistent estimator for $\sigma_{P,d}(\ell)$. See Algorithm 1, below, for its construction. To ensure uniform convergence of the variance weighted processes, we weigh $\hat{\nu}_d(\ell)$ by a trimmed version of the standard error estimators, $\hat{\sigma}_{d,\xi}(\ell) = \max\{\xi, \hat{\sigma}_d(\ell)\}$, where $\xi > 0$ is a trimming constant chosen by the user. See footnote 9 for the choice of ξ in our simulation study. We then define a

Kolmogorov-Smirnov (KS) type test statistic,

$$\widehat{S}_n = \sup_{d\in\{0,1\},\ \ell\in\mathcal{L}} \frac{\sqrt{nh}\cdot \hat{v}_d(\ell)}{\hat{\sigma}_{d,\xi}(\ell)}. \tag{6}$$

A large value of $\widehat{S}_n$ is statistical evidence against the null hypothesis. The cardinality of $\mathcal{L}$ is infinite if Y is continuously distributed, while with our construction of $\hat{v}_d(\ell)$ and $\hat{\sigma}_{d,\xi}(\ell)$ shown in Appendix A, we can coarsen $\mathcal{L}$ to the class of intervals spanned by the observed values of Y in the sample,

$$\hat{\mathcal{L}} \equiv \left\{[Y_i, Y_j] : Y_i \le Y_j, i, j \in \{1, \dots n\}\right\}, \tag{7}$$

without changing the value of the test statistic. In the Monte Carlo studies of Section 4 and the empirical applications of Section 5, we standardize and rescale the range of Y to the unit interval (by applying a transformation through the cdf of the standard normal distribution $\Phi(\cdot)$),[8] and employ the following coarsening of the class of intervals:

$$\mathcal{L}_{coarse} = \Big\{(y, y+c) :\ c^{-1} = q, \text{ and } q\cdot y \in \{0, 1, 2, \cdots, (q-1)\} \text{ for } q = 1, 2, \cdots, Q\Big\}. \tag{8}$$

As done in Hansen (1996), and Barrett and Donald (2003) in different contexts, we obtain asymptotically valid critical values by approximating the null distribution of the statistic using multiplier bootstrap. Algorithm 1, below, summarizes the implementation of our test. Theorems D.1-D.3 in Appendix D of the Supplemental Material show that the proposed test controls size at pre-specified significant levels uniformly, rejects fixed alternatives with probability approaching one, and has good power against a class of local alternatives.

Algorithm 1. (Implementation)

i. Specify a finite class of intervals $\mathcal{L}^*$. For instance, $\mathcal{L}^* = \hat{\mathcal{L}}$ of (7), or a coarsened version with the standardized outcome, $\mathcal{L}^* = L_{coarse}$ of (8) with a choice of finite integer Q (e.g., $Q = 15$).

ii. For each $\ell \in \mathcal{L}^*$, let $\hat{m}_{1,+}(\ell)$ and $\hat{m}_{1,-}(\ell)$ be local linear estimators for $\lim_{r\downarrow r_0} \mathbb{E}_P[g_\ell(Y)D|R = r]$ and $\lim_{r\uparrow r_0} \mathbb{E}_P[g_\ell(Y)D|R = r]$, respectively. Similarly, let $\hat{m}_{0,+}(\ell)$ and $\hat{m}_{0,-}(\ell)$ be local linear estimators for $\lim_{r\downarrow r_0} \mathbb{E}_P[g_\ell(Y)(1-D)|R = r]$ and $\lim_{r\uparrow r_0} \mathbb{E}_P[g_\ell(Y)(1-D)|R = r]$,

[8]Since the null hypothesis and the test statistic are invariant to strictly monotonic transformations of Y, this standardization does not affect the theoretical guarantee and the empirical results of our test.

respectively. See equation (11) in Appendix A for their closed-form expressions. Obtain $\hat{\nu}_1(\ell)$ and $\hat{\nu}_0(\ell)$ as follows:

$$\hat{\nu}_1(\ell) = \hat{m}_{1,-}(\ell) - \hat{m}_{1,+}(\ell), \quad \hat{\nu}_0(\ell) = \hat{m}_{0,+}(\ell) - \hat{m}_{0,-}(\ell). \tag{9}$$

iii. For each $\ell \in \mathcal{L}^*$, calculate sample analogs of the influence functions

$$\hat{\phi}_{\nu_1,i}(\ell) = \sqrt{nh}\left(w^-_{n,i} \cdot (g_\ell(Y_i)D_i - \hat{m}_{1,-}(\ell)) - w^+_{n,i} \cdot (g_\ell(Y_i)D_i - \hat{m}_{1,+}(\ell))\right),$$
$$\hat{\phi}_{\nu_0,i}(\ell) = \sqrt{nh}\left(w^+_{n,i} \cdot (g_\ell(Y_i)(1-D_i) - \hat{m}_{0,+}(\ell)) - w^-_{n,i} \cdot (g_\ell(Y_i)(1-D_i) - \hat{m}_{0,-}(\ell))\right),$$

where the definitions of the weighting terms $\{(w^+_{n,i}, w^-_{n,i}) : i = 1, \ldots, n\}$ are given in Appendix A. We then estimate the asymptotic standard deviation $\sigma_{P,d}(\ell)$ by $\hat{\sigma}_d(\ell) = \sqrt{\sum_{i=1}^n \hat{\phi}^2_{\nu_d,i}(\ell)}$ and obtain the trimmed estimators as $\hat{\sigma}_{d,\xi}(\ell) = \max\{\xi, \hat{\sigma}_d(\ell)\}$.[9]

iv. Calculate the test statistic $\widehat{S}_n = \widehat{S}_n = \sup_{d\in\{0,1\},\ \ell\in\mathcal{L}^*} \frac{\sqrt{nh}\cdot\hat{\nu}_d(\ell)}{\hat{\sigma}_{d,\xi}(\ell)}$.

v. Let a_n and B_n be sequences of non-negative numbers. For $d = 0, 1$ and $\ell \in \mathcal{L}$, define $\psi_{n,d}(\ell)$ as

$$\psi_{n,d}(\ell) = -B_n \cdot 1\left\{\frac{\sqrt{nh}\cdot\hat{\nu}_d(\ell)}{\hat{\sigma}_{d,\xi}(\ell)} < -a_n\right\}. \tag{10}$$

Following Andrews and Shi (2013, 2014), we use $a_n = \sqrt{0.3\ln(n)}$ and $B_n = \sqrt{\frac{0.4\ln(n)}{\ln\ln(n)}}$.

vi. Draw $U_1, U_2, \cdots U_n$ as i.i.d. standard normal random variables that are independent of the original sample. Compute the bootstrapped processes, $\widehat{\Phi}_{\nu_1}(\ell)$ and $\widehat{\Phi}_{\nu_0}(\ell)$, defined as

$$\widehat{\Phi}_{\nu_1}(\ell) = \sum_{i=1}^n U_i \cdot \hat{\phi}_{\nu_1,i}(\ell), \quad \widehat{\Phi}_{\nu_0}(\ell) = \sum_{i=1}^n U_i \cdot \hat{\phi}_{\nu_0,i}(\ell).$$

vii. Iterate Step (vi) $\bar{B}$ times ($\bar{B}$ is a large integer) and denote the realizations of the bootstrapped processes by $\left(\widehat{\Phi}^b_{\nu_1}(\cdot), \widehat{\Phi}^b_{\nu_0}(\cdot) : b = 1, \ldots, \bar{B}\right)$. Let $\hat{q}(\tau)$ be the τ-th empirical quantile of $\left\{\sup_{d\in\{0,1\},\ \ell\in\mathcal{L}^*}\left\{\frac{\widehat{\Phi}^b_{\nu_d}(\ell)}{\hat{\sigma}_{d,\xi}(\ell)} + \psi_{n,d}(\ell)\right\} : b = 1, \ldots, \bar{B}\right\}$. For significance level $\alpha < 1/2$, obtain

[9] In the simulations, we set $\xi = \sqrt{a(1-a)}$, where $a = 0.0001$. We also use $a \in \{0.001, 0.03, 0.5\}$. The results are insensitive to the choice of a. These tuning parameters are motivated by the observation that the denominator of the asymptotic variance takes the form of $p_\ell(1-p_\ell)$, where $p_\ell = \lim_{r\to r_0} \mathbb{P}(Y \in C_\ell, D = d | R = r)$.

a critical value $\hat{c}_{\eta}(\alpha)$ of the test by $\hat{c}_{\eta}(\alpha) = \hat{q}(1-\alpha+\eta)+\eta$, where $\eta > 0$ is an arbitrarily small positive number, e.g., 10^{-6}.[10]

viii. Reject H_0 if $\widehat{S}_n > \hat{c}_{\eta}(\alpha)$.

Following the existing papers in the moment inequality literature, Step vii in Algorithm 1 uses the generalized moment selection (GMS) proposed by Andrews and Soares (2010), and Andrews and Shi (2013). It is similar to the recentering method of Hansen (2005), and Donald and Hsu (2016), and the contact set approach of Linton, Song, and Whang (2010).

Regarding the bandwidths for the local linear estimators in step ii, our informal recommendation is to have the bandwidth of $\hat{m}_{d,+}(\ell)$, $d = 1, 0$, common for all $\ell \in \mathcal{L}^*$ and the bandwidth of $\hat{m}_{d,-}(\ell)$, $d = 1, 0$, common for all $\ell \in \mathcal{L}^*$. We denote the two bandwidths by h_+ and h_-, respectively, and allow $h_+ \neq h_-$. There is merit to using the bandwidths that are recommended for point estimation of the LATE at the cut-off, such as the bandwidths suggested in Imbens and Kalyanaraman (2012), Calonico, Cattaneo, and Titiunik (2014), and Arai and Ichimura (2016). This is because the FRD-Wald estimator is numerically equal to the difference of the means between the following distribution function estimates for compliers:

$$\begin{aligned}
\hat{F}_{Y_1(r_0)|\mathbf{C},R=r_0}(y) &= \frac{\hat{m}_{1,+}((-\infty,y)) - \hat{m}_{1,-}((-\infty,y))}{\hat{\pi}^+ - \hat{\pi}^-}, \\
\hat{F}_{Y_0(r_0)|\mathbf{C},R=r_0}(y) &= \frac{\hat{m}_{0,-}((-\infty,y)) - \hat{m}_{0,+}((-\infty,y))}{\hat{\pi}^+ - \hat{\pi}^-},
\end{aligned}$$

where $\hat{m}_{1,+}((-\infty,y))$ and $\hat{m}_{0,+}((-\infty,y))$ use h_+, $\hat{m}_{1,-}((-\infty,y))$ and $\hat{m}_{0,-}((-\infty,y))$ use h_-, and $\hat{\pi}^+$ and $\hat{\pi}^-$ are the local linear estimators for $\lim_{r\downarrow r_0} P(D=1|R=r)$ and $\lim_{r\uparrow r_0} P(D=1|R=r)$ with bandwidths h_+ and h_-, respectively. Accordingly, reusing these bandwidths to compute our test statistic, we assess nonnegativity of the compliers' potential outcome densities based on the same in-sample information as that which the point estimate for the compliers' causal effect relies on.[11]

[10] This η constant is called an infinitesimal uniformity factor and is introduced by Andrews and Shi (2013) to avoid the problems that arise due to the presence of the infinite-dimensional nuisance parameters $\nu_{P,1}(\ell)$ and $\nu_{P,0}(\ell)$.

[11] Alternatively, we may want to choose bandwidths so as to optimize a power criterion. We leave power-optimizing choices of bandwidth for future research. Algorithm 1 provides some default choices for other tuning parameters, ξ, a_n, and B_n, without claiming that these choices are optimal. According to our Monte Carlo studies and empirical applications considered in Sections 4 and 5, the test results are not sensitive to mild departures from the default choices.

4. SIMULATION

This section investigates the finite sample performance of the proposed test by Monte Carlo experiments. We consider six data generating processes (DGPs) including two DGPs, Size1-Size2, for examining the size properties and four DGPs, Power1-Power4, for examining the power properties of the test. For all DGPs, we set the cut-off point at $r_0 = 0$.

4.1. Size properties.

Size1 Let $R \sim N(0,1)$ truncated at -2 and 2. The propensity score $P(D=1|R=r) = 0.5$ for all r. $Y|(D=1, R=r) \sim N(1,1)$ for all r and $Y|(D=0, R=r) \sim N(0,1)$ for all r.

Size2 Same as Size1 except that

$$P(D=1|R=r) = \mathbf{1}\{-2 \leq r < 0\}\frac{(r+2)^2}{8} + \mathbf{1}\{0 \leq r \leq 2\}\left(1 - \frac{(r-2)^2}{8}\right).$$

In both DGPs, the propensity scores are continuous at the cut-off (i.e., Assumption 3 does not hold). Combined with FRD-validity (Assumptions 1 and 2), the distributions of the observables are also continuous at the cut-off, implying that these DGPs correspond to least favorable nulls in the context of our test. Size1 has a constant propensity score, while in Size2, the left- and right-derivatives of the propensity scores differ at the cut-off.

For each DGP, we generate random samples of four sizes: 1000, 2000, 4000, and 8000 observations. We specify $\mathcal{L}^* = \mathcal{L}_{coarse}$ with $Q = 15$.[12] For each simulation design, we conduct 1000 repetitions with $\bar{B} = 300$ bootstrap iterations. We consider three data-driven choices of bandwidths: Imbens and Kalyanaraman (2012, IK), Calonico, Cattaneo, and Titiunik (2014, CCT), and Arai and Ichimura (2016, AI). For each bandwidth, we impose undersmoothing by multiplying $n^{\frac{1}{5}-\frac{1}{c}}$ and the bandwidth, choosing $c = 4.5$.[13] In addition, we also consider the MSE-optimal robust bias correction (MSE-RBC) implementation (see Calonico, Cattaneo, and Farrell, 2018) and the coverage error rate-optimal (CER-RBC) implementation (see Calonico, Cattaneo, and Farrell, 2020). For the MSE-RBC bandwidth, we implement the test by estimating the conditional means via local quadratic regression using a bandwidth that is MSE-optimal for local linear regression (AI, IK, or CCT), see Calonico, Cattaneo, and Titiunik (2014, Remark 7). For the CER-RBC bandwidth, we multiply a MSE-optimal

[12] We note that our test exhibits similar results when Q is greater than 10.

[13] We run simulations for other choices of the under-smoothing constant $c \in [3, 5)$; the results are similar.

bandwidth (AI, IK, or CCT) by the rule-of-thumb adjustment factor proposed in Calonico, Cattaneo, and Farrell (2020, section 4).

TABLE 1. Rejection Frequency at the 5% Level

DGP	n	US AI	US IK	US CCT	MSE-RBC AI	MSE-RBC IK	MSE-RBC CCT	CER-RBC AI	CER-RBC IK	CER-RBC CCT
Size1	1000	0.060	0.02	0.019	0.037	0.016	0.017	0.055	0.025	0.024
	2000	0.071	0.025	0.034	0.058	0.018	0.022	0.076	0.022	0.024
	4000	0.078	0.038	0.035	0.067	0.049	0.022	0.086	0.043	0.035
	8000	0.065	0.045	0.033	0.066	0.051	0.037	0.057	0.045	0.036
Size 2	1000	0.063	0.014	0.012	0.039	0.008	0.015	0.067	0.015	0.014
	2000	0.064	0.035	0.031	0.051	0.033	0.021	0.060	0.032	0.024
	4000	0.064	0.041	0.038	0.077	0.040	0.037	0.060	0.042	0.039
	8000	0.060	0.039	0.036	0.065	0.042	0.040	0.057	0.044	0.035

Table 1 summarizes the results at the 5% nominal level. For the full set of results at other significance levels, see Tables F.1 to F.3 in Appendix F of the Supplemental Material. The results show that the proposed test controls size well for each of the specified designs and the various bandwidth choices. Although our test statistic (which takes the supremum over a class of intervals) is different from those considered in Calonico, Cattaneo, and Farrell (2020), and Calonico, Cattaneo, and Farrell (2020), the CER-RBC and MSE-RBC implementations work well.

4.2. **Power properties.** To investigate the power properties, we consider the following four DGPs, Power1-Power4, in which the conditional distribution of Y_1 violates the local continuity condition in different ways.[14]

Power1 Let $R \sim N(0,1)$ truncated at -2 and 2. The propensity score is given by

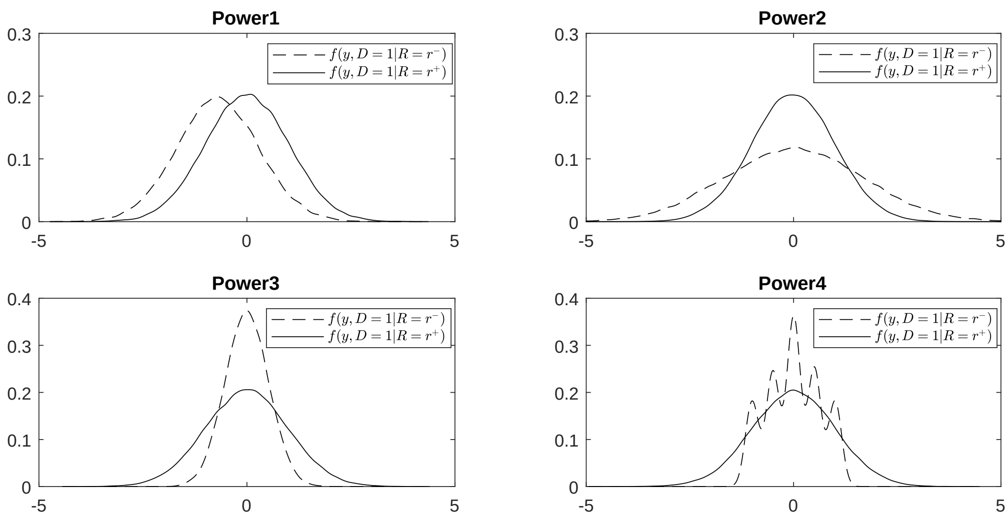
$$P(D=1|R=r) = \mathbf{1}\{-2 \leq r < 0\} \max\{0, (r+2)^2/8 - 0.01\} + \mathbf{1}\{0 \leq r \leq 2\} \min\{1, 1-(r-2)^2/8 + 0.01\}$$

Let $Y|(D=0, R=r) \sim N(0,1)$ for all $r \in [-2,2]$, and $Y|(D=1, R=r) \sim N(0,1)$ for all $r \in [0,2]$. Let $Y|(D=1, R=r) \sim N(-0.7,1)$ for all $r \in [-2,0)$.

Power2 Same as Power1 except that $Y|(D=1, R=r) \sim N(0, 1.675^2)$ for all $r \in [-2,0)$.

[14] Appendix F of the Supplemental Material provide examples where violation of the local monotonicity assumption or the local continuity assumption results in distributions of observables similar to those for Power1.

FIGURE 1. Potential Outcome Densities at the Cut-off



Power3 Same as Power1 except that $Y|(D=1, R=r) \sim N(0, 0.515^2)$ for all $r \in [-2, 0)$.

Power4 Same as Power1 except that $Y|(D=1, R=r) \sim \sum_{j=1}^{5} \omega_j N(\mu_j, 0.125^2)$ for all $r \in [-2, 0)$, where $\omega = (0.15, 0.2, 0.3, 0.2, 0.15)$ and $\mu = (-1, -0.5, 0, 0.5, 1)$.

Figure 1 plots the potential outcome density at the cut-off for each of Power1-Power4, in which the testable implication of Theorem 1 (i) is violated since the solid curves and the dashed curves intersect. Table 2 reports simulation results for the power properties of our test at the 5% level. Additional results are collected in Tables F.4 to F.6 in Appendix F. Overall, our test has good power in detecting deviations from the null under all choices of bandwidth. It is harder for our test to reject in Power4. From Figure 1, we see that the violation of the null in Power4 occurs abruptly with many peaks over narrow intervals, whereas in the other designs (e.g., Power1 and Power2) mild violation occurs over relatively wide intervals. This phenomenon is consistent with what has been noted in the literature: the Bierens (1982)-, and Andrews and Shi (2013)-type methods that we adopt in this paper are efficient in detecting the second type of violations.[15]

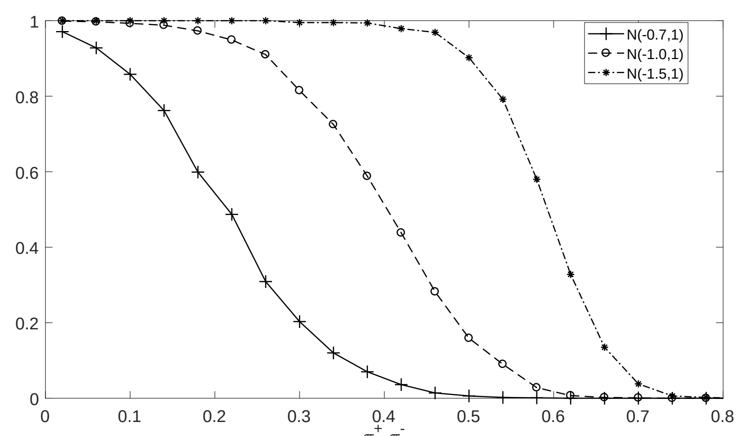
As the magnitude of the propensity score jump $\pi^+ - \pi^-$ becomes smaller, we expect that the inequalities of (2) and (3) become closer to binding. For instance, in the extreme case of

[15] See Chernozhukov, Lee, and Rosen (2013, footnote 10) for related discussion.

TABLE 2. Rejection Frequency at the 5% Level

DGP	n	US AI	US IK	US CCT	MSE-RBC AI	MSE-RBC IK	MSE-RBC CCT	CER-RBC AI	CER-RBC IK	CER-RBC CCT
Power1	1000	0.215	0.174	0.111	0.103	0.081	0.050	0.225	0.143	0.090
	2000	0.439	0.403	0.256	0.221	0.207	0.115	0.439	0.316	0.205
	4000	0.753	0.744	0.604	0.476	0.459	0.305	0.749	0.629	0.486
	8000	0.962	0.975	0.907	0.812	0.820	0.654	0.964	0.935	0.831
Power2	1000	0.122	0.061	0.052	0.086	0.023	0.022	0.133	0.046	0.045
	2000	0.271	0.194	0.140	0.135	0.099	0.052	0.266	0.156	0.097
	4000	0.554	0.511	0.342	0.293	0.246	0.142	0.560	0.399	0.248
	8000	0.885	0.888	0.732	0.624	0.622	0.391	0.889	0.793	0.598
Power3	1000	0.164	0.123	0.078	0.106	0.063	0.027	0.159	0.107	0.061
	2000	0.299	0.257	0.170	0.174	0.154	0.079	0.306	0.209	0.128
	4000	0.573	0.510	0.383	0.361	0.289	0.183	0.581	0.421	0.321
	8000	0.883	0.870	0.734	0.694	0.640	0.466	0.888	0.781	0.640
Power4	1000	0.099	0.050	0.024	0.057	0.024	0.017	0.101	0.036	0.027
	2000	0.172	0.123	0.060	0.118	0.060	0.042	0.175	0.092	0.074
	4000	0.264	0.268	0.144	0.181	0.144	0.079	0.265	0.201	0.138
	8000	0.550	0.540	0.326	0.341	0.326	0.201	0.545	0.438	0.283

FIGURE 2. Power and Propensity Jump Size



$\pi^+ - \pi^- = 0$, for a distribution satisfying the testable implication, inequalities (2) and (3) must hold with equality, i.e., the conditional distribution of $(Y, D)|R$ is continuous at the cut-off. This means a joint distribution of potential outcomes and selection type violating FRD-validity is more likely to violate the testable implications as the magnitude of the jump in the propensity score becomes smaller. In the opposite direction, the testable implication of Theorem 1 loses screening power when the FRD design is close to a sharp design.

We illustrate this point by modifying the propensity score of Power1 to

$$P(D=1|R=r) = \mathbf{1}\{-2 \le r < 0\} \max\{0, (r+2)^2/8 - d\} \\ + \mathbf{1}\{0 \le r \le 2\} \min\{1, 1-(r-2)^2/8 + d\}.$$

Here, $2d$ measures the jump size of the propensity score and $d = 0.01$ corresponds to the results of Power1. In addition to the specification $Y|(D=1, R=r) \sim N(-0.7, 1)$ for $r \in [-2, 0)$, we consider two additional specifications, $Y|(D=1, R=r) \sim N(-1, 1)$ and $Y|(D=1, R=r) \sim N(-1.5, 1)$ for $r \in [-2, 0)$, which lead to larger deviations from the null.

Figure 2 plots the rejection frequency as a function of $\pi^+ - \pi^- = 2d$ for each of the alternative distributions at the 5% level for a sample size of 8000 observations.[16] At each specification of $Y|(D=1, R=r)$ for $r \in [-2, 0)$, we see that the rejection frequency decreases as the jump size increases. As the jump size approaches one (the sharp design), the rejection frequency falls to zero because inequalities (2) and (3) are never violated in the sharp design. On the other hand, for a given jump size, a larger deviation from local continuity leads to a larger rejection frequency, as expected.

5. Applications

To illustrate that implementing our test can provide new insights for empirical practice, we assess FRD-validity in the designs studied in Angrist and Lavy (1999, AL hereafter), and Miller, Pinto, and Vera-Hernández (2013, MPV hereafter).

5.1. **Effect of class size on student performance.**

Israel has been implementing Maimonides' rule in public schools since 1969. The rule limits a class size to 40 students and so creates discontinuous changes in the average class size as the total enrollment exceeds multiples of 40 students. For example, a public school with 40 enrolled students in a grade can maintain one class, with a (average) class size of 40 students; another public school with 41 enrolled students has to offer two classes, and so the average class size drops discontinuously from 40 students to 20.5 students. Maimonides' rule

[16] Here we only report the results based on the under-smoothed IK bandwidth. Other choices produce similar results.

offers an example of FRD design since some schools in the data do not comply with the treatment assignment rule.[17]

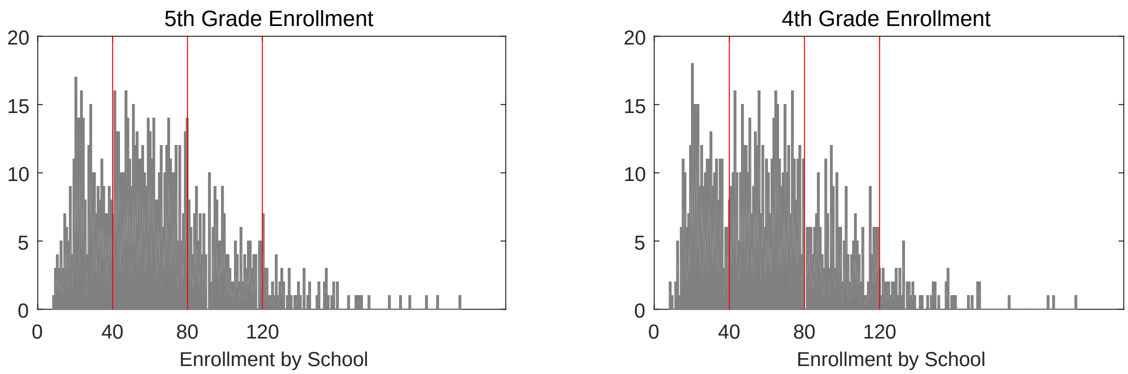
Recent empirical evidence suggests that the density of the running variable (enrollment) is discontinuous near some cut-offs (Otsu, Xu, and Matsushita (2013), and Angrist, Lavy, Leder-Luis, and Shany (2019)). Along with the argument of Lee (2008), and McCrary (2008), this evidence raises concerns about FRD-validity, but cannot be interpreted as direct evidence to refute local continuity or local monotonicity.

Who manipulates class size? As argued in AL, parents may selectively exploit Maimonides' rule by either (a) registering their children into schools with enrollments slightly above multiples of 40 students, hoping that their children will be placed in smaller classes, or (b) withdrawing children from those public schools with enrollments slightly below multiples of 40 students. In either case, we expect to observe discontinuities of the density of the running variable at the cut-offs, as we can observe most notably at the enrollment count of 40 students in Figure 3. Class size manipulation by parents can be a serious threat to the local continuity assumption if those parents who act according to (a) also more highly value a small class-size education, and are more concerned with their children's education. If children with such parents perform better than their peers, the potential outcome distributions of the students' test scores violate local continuity.

On the other hand, AL defend FRD-validity by arguing that manipulation of class size by parents is not likely. Concerning the possibility of (a), AL claim that: "*there is no way [for the parents] to know [exactly] whether a predicted enrollment of 41 [students] will not decline to 38 [students] by the time school starts, obviating the need for two small classes*". With respect to the possibility of (b), private elementary schooling is rare in Israel and withdrawing their children is not a feasible option for most parents. Angrist, Lavy, Leder-Luis, and Shany (2019) re-investigate Maimonides' rule and argue that the manipulation is operated mainly on the school board side, stating that: "*A recent memo from Israeli Ministry of Education (MOE) officials to school leaders admonishes headmasters against*

[17] We define the treatment as whether the school "splits" ($D = 1$) or does "not split" ($D = 0$) a cohort with an enrollment around the cut-off into smaller classes. Focusing on grade 4, with bandwidth equal to 3 and the cut-off at 80 students as an example, we first restrict the sample to classes if their schools' grade 4 enrollment is $R \in \{78, 79, 80\} \cup \{81, 82, 83\}$ students. Then we assign $D = 1$ to a class if its school has "three classes" and assign $D = 0$ to a class if its school has "two classes". In the data, there are schools that have an enrollment within $\{78, 79, 80\} \cup \{81, 82, 83\}$ students but that have either one or more than three classes. They are very rare (about 0.2% of the total observations), and we exclude these observations from our analysis.

FIGURE 3. Histograms for Enrollments by Schools: Panel A of Figure 6 in Angrist, Lavy, Leder-Luis, and Shany (2019)



attempts to increase staffing ratios through enrollment manipulation. In particular, schools are warned not to move students between grades or to enroll those who are overseas so as to produce an additional class." This type of manipulation can lead to a density jump like that observed in Figure 3, but is not necessarily a serious threat to FRD-validity depending on the school board's incentives to manipulate. If the main motivation of manipulation is to increase their budget (an increasing function of the number of classes), as argued in Angrist, Lavy, Leder-Luis, and Shany (2019), and if the distributions of the students' potential outcomes in those schools where boards manipulate enrollment are the same as those in schools where boards do not manipulate, any manipulation around the cut-off is independent of the students' unobserved talents. Then, FRD-validity can hold even when the density of the running variable is discontinuous at the cut-offs.

Test Results. The testable implication assessed by our test focuses on the joint distribution of the observed outcomes and treatment status, in contrast to the density continuity approach that focuses only on the marginal distribution of the running variable. Hence, our test can provide new empirical evidence that can contribute to the dispute about the FRD-validity of Maimonides' rule, reviewed above.

We apply the test proposed in Section 3 for each of the four outcome variables (grade 4 math and verbal test scores, and grade 5 math and verbal test scores) by treating the three cut-offs of 40, 80, and 120 students, separately. We consider the bandwidths ($h_+ = h_- = 3$ and $h_+ = h_- = 5$)

used in AL, as well as the three data-driven bandwidth choices (AI, IK and CCT).[18] We also report p-values using the RBC bandwidth choices based on CCT with MSE-Optimal and CER-Optimal criteria, respectively. We set the trimming constant to $\xi = 0.00999$, as described in Algorithm 1 of Section 3.[19]

Table 3 displays the p-values of the tests. For all the cases considered, we do not reject the null hypothesis at a 10% significance level. The results are robust to the choice of bandwidths and the choice of trimming constants (see Tables G.2 to G.4 in Appendix G of the Supplemental Material). Despite the fact that the density of the running variable appears to be discontinuous at the cut-off, "no rejection" by our test suggests empirical support for the argument of "manipulation by the school board"–the type of manipulation that is relatively innocuous for AL's identification strategy. As discussed in Section 4.2 and illustrated in Figure 2, it is, however, important to acknowledge that the statistical power of our test might be limited by the large jumps in the propensity score that occur in this application, ranging from 0.3 to 0.7 (see Table G.1 in the Supplemental Material).

5.2. **Colombia's Subsidized Regime.** MPV study the impact of "Colombia's Régimen Subsidiado (SR)," a publicly financed insurance program targeted at poor households, on financial risk protection, service use, and health outcomes. SR subsidizes eligible Colombians to purchase insurance from private and government-approved insurers. Program eligibility is determined by a threshold rule based on a continuous index called Sistema de Identificacion de Beneficiarios (SISBEN) ranging from 0 to 100 (with 0 being the most impoverished, and those below a cut-off being eligible). SISBEN is constructed by a proxy means-test using fourteen different measurements of a household's well-being. It is, however, well known that the original SISBEN index used to assign the actual program eligibility was manipulated by either households or the administering authority (see MPV and the references therein for details). To circumvent this issue of manipulation, MPV simulate their own SISBEN index for each household using a collection of survey data from independent sources. MPV then estimate a cut-off of the simulated SISBEN scores in each region by maximizing the performance of in-sample prediction for the actual program take-up. Using these estimated cut-offs, MPV estimate the compliers' effects of SR on 33 outcome variables in four categories: (i) risk

[18] See Table G.11 in Appendix G of the Supplemental Material for the obtained bandwidths and the number of observations therein.

[19] We try different choices for the trimming constant $\xi \in \{0.0316, 0.1706, 0.5\}$ and obtain similar results.

TABLE 3. Testing Results for Israeli School Data: p-values, $\xi = 0.00999$

	3	5	AI	IK	CCT	MSE-RBC	CER-RBC
g4math							
Cut-off 40	0.986	0.934	0.767	0.978	0.968	0.964	0.975
Cut-off 80	0.909	0.865	0.715	0.944	0.888	0.771	0.957
Cut-off 120	0.443	0.702	0.665	0.604	0.568	0.613	0.639
g4verb							
Cut-off 40	0.928	0.627	0.465	0.648	0.529	0.564	0.463
Cut-off 80	0.911	0.883	0.185	0.906	0.720	0.284	0.842
Cut-off 120	0.935	0.683	0.474	0.730	0.186	0.228	0.143
g5math							
Cut-off 40	0.876	0.282	0.488	0.631	0.609	0.901	0.265
Cut-off 80	0.516	0.446	0.930	0.482	0.765	0.808	0.726
Cut-off 120	0.939	0.827	0.626	0.883	0.838	0.842	0.772
g5verb							
Cut-off 40	0.594	0.893	0.953	0.906	0.938	0.955	0.962
Cut-off 80	0.510	0.692	0.504	0.525	0.929	0.953	0.973
Cut-off 120	0.696	0.811	0.601	0.699	0.781	0.739	0.745

protection, consumption smoothing, and portfolio choice, (ii) medical care use, (iii) health status, and (iv) behavior distortions; see Table G.5 of the Supplemental Material and Table 1 of MPV for details.

Although the density of the simulated SISBEN score passes the continuity test (see MPV's online Appendix C), it does not necessarily imply FRD-validity, e.g., the conditional distributions of the potential outcomes given the simulated SISBEN score may not be continuous at the cut-off.

For each of the 33 outcome variables, we implement our test using MPV's simulated SISBEN score as the running variable and the actual program enrollment as the treatment status. We consider the three bandwidths ($h_+ = h_- = 2, 3,$ and 4) used in MPV as well as the three data-driven bandwidth choices (AI, IK and CCT).[20] We use the same set of trimming constants ξ as in the AL application and find that the results are insensitive to a choice of ξ. We find robust evidence to reject the testable implications of FRD-validity for the following three outcome variables: "household education spending," "total spending on food," and "total monthly expenditure." Their p-values are

[20]See Table G.12 in Appendix G of the Supplemental Material for the obtained bandwidths and the number of observations contained therein.

reported in Table 4 (results for all other outcome variables and other choices of ξ are collected in Tables G.5-G.8 in Appendix G of the Supplemental Material).

TABLE 4. Testing Results for Columbia's SR Data: p-values ($\xi = 0.00999$)

Outcome variables	MPV Bandwidths 2	3	4	Other Bandwidth Choices AI	IK	CCT	MSE RBC	CER RBC
Household education spending	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00
Total spending on food	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00
Total monthly expenditure	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

A few remarks are in order. First, the three outcome variables giving the robust rejections all belong to the first category: "risk protection, consumption smoothing, and portfolio choice." For other outcome variables, we do not find evidence against FRD-validity. The low p-values for these three outcomes remain significant even when we take into account multiple-testing of a group of outcome variables with family-wise error rate (FWER) control.[21]

Second, it is possible to figure out which observations cause the rejection of FRD-validity. Take the choice of $\mathcal{L}^* = \mathcal{L}_{coarse}$ with $Q = 15$ and bandwidth $h^+ = h^- = 2$, and focus on the outcome variable "total spending on food" as an example. The supremum in the test statistic is achieved at $d = 0$ and $[y, y'] = [0.5948, 0.6527]$. Figure 4 draws the kernel smoothed (pseudo) densities of the normalized outcome variable, where the blue curve should be underneath the red curve under FRD-validity. The two density curves in the top-right panel indeed cross in this interval. The histograms and Table 5 show that there are 45 observations with $D = 0, R \in (r_0 - 2, r_0)$ and $Y \in [0.5948, 0.6527]$, which are about 6.73% of all observations with $R \in (r_0 - 2, r_0)$ and $D = 0$. On the other hand, there are 31 observations with $D = 0, R \in [r_0, r_0 + 2)$ and $Y \in [0.5948, 0.6527]$, which is about 2.06% of all observations with $R \in [r_0, r_0 + 2)$ and $D = 0$. See Figures G.1, G.2 and Tables G.13, G.14 in the Supplemental Material for similar analysis on the other two outcome variables in Table 4.

Third, we also condition on each of the six regions in Colombia when implementing the test. The results are collected in Table G.9 in the Supplemental Material. We obtain strong rejections

[21]The results shown in Table G.5 of Appendix G of the Supplemental Material imply that, for the first category of 10 outcome variables, the multiple testing procedure of Holm (1979) concludes that the joint null hypothesis of FRD-validity holding for the 10 outcomes is rejected for the control of FWER at 1%. With all the outcomes (33 hypotheses), the joint null hypothesis is rejected for the control of FWER at 5%.

TABLE 5. Obs. in the maximizer interval ($h^+ = h^- = 2$): Total spending on food

	# of observations		
Subsample of	All	$\{0.5948 \le Y \le 0.6527\}$	Ratio
$\{0 \le R < h^+\} \cap \{D = 0\}$ ($\mathbf{N} \cup \mathbf{C}$)	1502	31	2.06%
$\{h^- < R < 0\} \cap \{D = 0\}$ ($\mathbf{N}$)	669	45	6.73%

in the "Atlantica", "Oriental", "Central", and "Bogota" regions, and no rejection in "Pacifico" and "Territorios Nacionales". Taking into account the relative sample sizes across the regions (Table G.10 of the Supplemental Material), the Bogota sample seems to drive the test results of Table 4. Notice that the magnitude of the propensity score jump for the Bogota sample is relatively small compared with the samples in the regions giving no-rejections (see Figure 5). This observation is in line with Figure 2 and the discussion of Section 4.

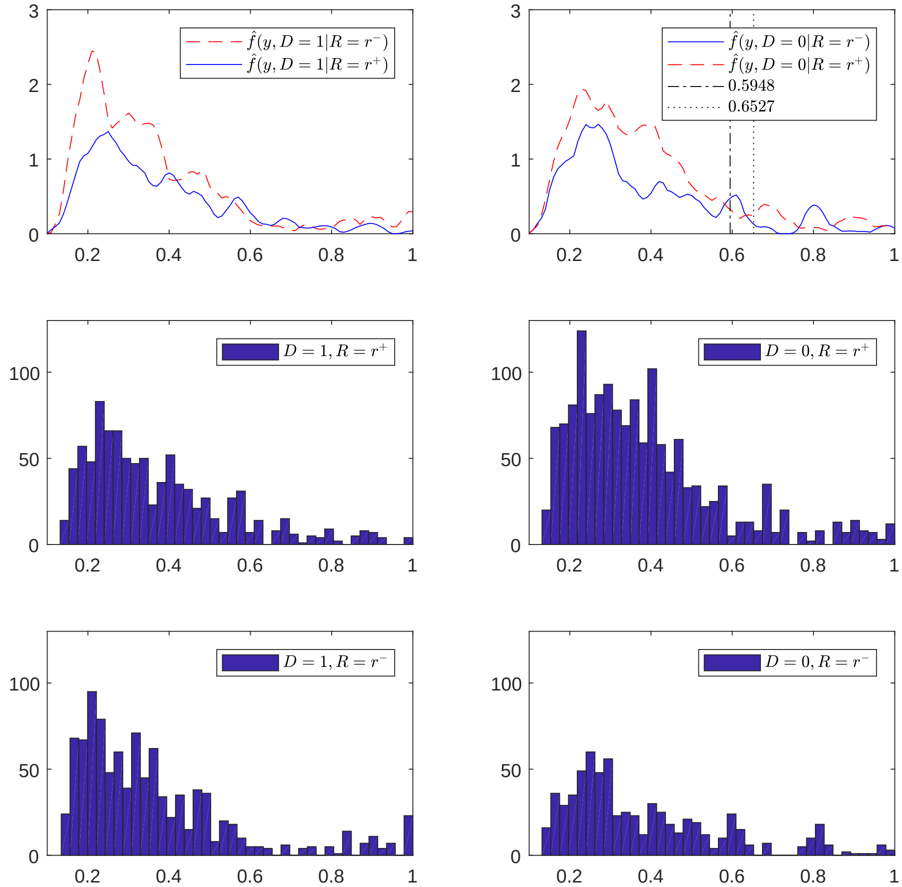
There are several possible reasons why FRD-validity fails in this application. First, violation of local continuity may arise as a byproduct of estimating the cut-off using the simulated SISBEN score. For instance, if there is some household characteristic that is *not* included in the construction of the simulated SISBEN score but has strong predictive power for program enrollment, the estimated cut-off may pick up a value of the simulated SISBEN score across which the distribution of the excluded characteristic differs most. If the distribution of household consumption variables depend on such an excluded characteristic, the result is a violation of local continuity. Second, there could be other *unobserved* programs using the same SISBEN index with similar cut-offs. If such programs significantly affect a household's budget, we can expect the distribution of potential household consumption to be quite different on each of the two sides of the cut-off, again leading to a violation of local continuity.[22]

6. CONCLUSION

In this paper we propose a specification test for the key identifying conditions in fuzzy regression design. We characterize the set of sharp testable implications for FRD-validity and propose an asymptotically valid test for it. Our approach makes use of not only the information conveyed by the

[22]MPV suggest that the second channel is less likely to be the cause of rejection of FRD-validity for the relevant three outcome variables. See Table 2 in MPV for evidence that the enrollment rates for other programs do not change across the estimated cut-offs.

FIGURE 4. Estimated complier's outcome density: Total spending on food



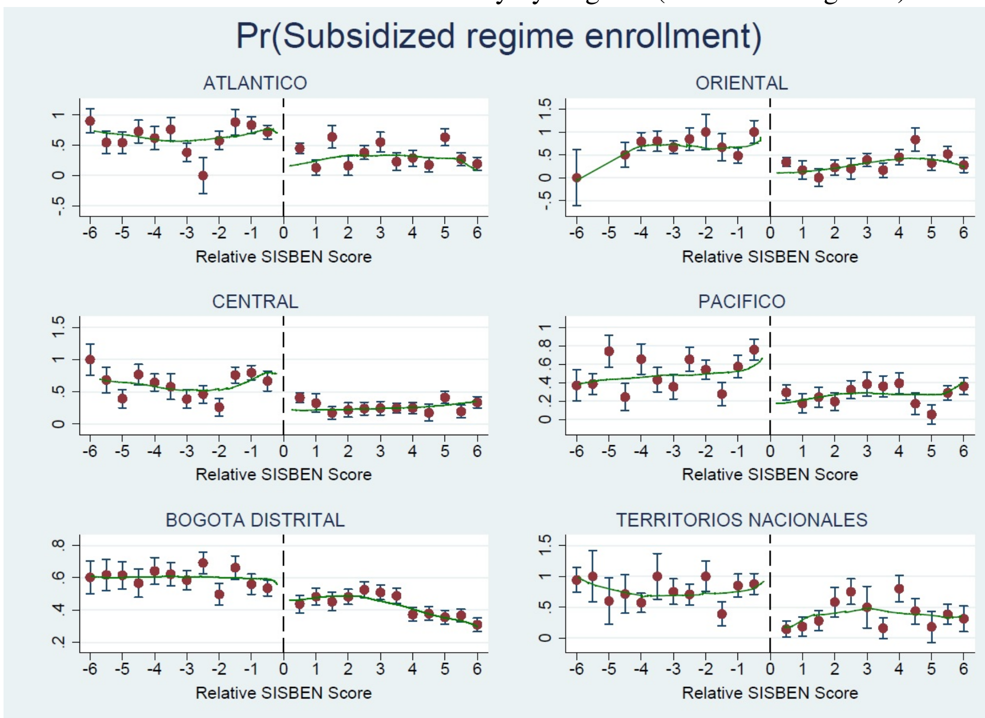
running variable but also that conveyed by the outcome and treatment status. As illustrated in our empirical applications, our specification test provides empirical evidence for or against FRD-validity, which is overlooked if only the continuity of the running variable's density at the cut-off is assessed.

APPENDIX A. CALCULATING THE TEST STATISTICS

We describe how to compute the proposed test statistic. Let $m_{P,d}(\ell, r) = \mathbb{E}_P[g_\ell(Y)D^d(1-D)^{1-d}|R=r]$ and $m_{P,d,+}(\ell) = \lim_{r\downarrow r_0} m_{P,d}(\ell, r)$ and $m_{P,d,-}(\ell) = \lim_{r\uparrow r_0} m_{P,d}(\ell, r)$ for $d = 1, 0$, then we can estimate $\nu_{P,1}(\ell)$ and $\nu_{P,0}(\ell)$ respectively by equation (9), which we restate below:

$$\hat{\nu}_1(\ell) = \hat{m}_{1,-}(\ell) - \hat{m}_{1,+}(\ell), \quad \hat{\nu}_0(\ell) = \hat{m}_{0,+}(\ell) - \hat{m}_{0,-}(\ell),$$

FIGURE 5. Enrollment Probability by Regions (from MPV Figure 2)



where the right hand side terms $\hat{m}_{d,\star}(\ell)$, for $d = 1, 0$ and $\star = +, -$, are local linear estimators. They can be constructed by the intercept estimates $\hat{a}_{d,+}(\ell)$ and $\hat{a}_{d,-}(\ell)$ in the local regressions,

$$
\begin{aligned}
&(\hat{a}_{d,+}(\ell), \hat{b}_{d,+}(\ell)) \\
&\quad = \underset{a,b}{\operatorname{argmin}} \frac{1}{nh_+} \sum_{i=1}^{n} 1\{R_i \geq r_0\} \cdot K\left(\frac{R_i - r_0}{h_+}\right) \left[g_\ell(Y_i) D_i^d (1 - D_i)^{1-d} - a - b \cdot \left(\frac{R_i - r_0}{h_+}\right)\right]^2, \\
&(\hat{a}_{d,-}(\ell), \hat{b}_{d,-}(\ell)) \\
&\quad = \underset{a,b}{\operatorname{argmin}} \frac{1}{nh_-} \sum_{i=1}^{n} 1\{R_i < r_0\} \cdot K\left(\frac{R_i - r_0}{h_-}\right) \left[g_\ell(Y_i) D_i^d (1 - D_i)^{1-d} - a - b \cdot \left(\frac{R_i - r_0}{h_-}\right)\right]^2,
\end{aligned}
$$

where $K(\cdot)$ is a kernel function and (h_+, h_-) are the bandwidths specified above and below the cut-off, respectively. In particular, we express $h_+ = c_+ h$ and $h_- = c_- h$, with (c_+, c_-) being positive constants and h being a sequence converging to zero as $n \to \infty$. For simplicity of analysis and implementation, we specify the bandwidths h_+ and h_- to be the same over $\{g_\ell : \ell \in \mathcal{L}\}$.

We can write the local linear estimators in the following form: for $d = 1, 0$ and $\star = +, -$

$$\hat{m}_{d,\star}(\ell) = \sum_{i=1}^{n} w_{n,i}^{\star} \cdot g_{\ell}(Y_i) D_i^d (1 - D_i)^{1-d}, \tag{11}$$

where the weights are defined as

$$w_{n,i}^{+} = \frac{1}{nh_+} \cdot \frac{1\{R_i \geq r_0\} \cdot K(\frac{R_i - r_0}{h_+}) \left[\hat{\vartheta}_2^{+} - \hat{\vartheta}_1^{+} \cdot \left(\frac{R_i - r_0}{h_+} \right) \right]}{\hat{\vartheta}_2^{+} \hat{\vartheta}_0^{+} - (\hat{\vartheta}_1^{+})^2},$$
$$w_{n,i}^{-} = \frac{1}{nh_-} \cdot \frac{1\{R_i < r_0\} \cdot K(\frac{R_i - r_0}{h_-}) \left[\hat{\vartheta}_2^{-} - \hat{\vartheta}_1^{-} \cdot \left(\frac{R_i - r_0}{h_-} \right) \right]}{\hat{\vartheta}_2^{-} \hat{\vartheta}_0^{-} - (\hat{\vartheta}_1^{-})^2}.$$

and for $j = 0, 1, 2$,

$$\hat{\vartheta}_j^{+} = \frac{1}{nh_+} \sum_{i=1}^{n} 1\{R_i \geq r_0\} \cdot K \left(\frac{R_i - r_0}{h_+} \right) \left(\frac{R_i - r_0}{h_+} \right)^j,$$
$$\hat{\vartheta}_j^{-} = \frac{1}{nh_-} \sum_{i=1}^{n} 1\{R_i < r_0\} \cdot K \left(\frac{R_i - r_0}{h_-} \right) \left(\frac{R_i - r_0}{h_-} \right)^j.$$

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SUPPLEMENT TO "TESTING IDENTIFYING ASSUMPTIONS IN FUZZY REGRESSION DISCONTINUITY DESIGNS"

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We describe how our test differs and complements existing tests in Appendix B. We discuss several extensions in Appendix C. We formally state the asymptotic validity of our test in Appendix D. All proofs are collected in Appendix E. Additional empirical results of Section 5 are provided in Appendix G.

Appendix B. Comparison between our approach and the existing approach

We provide a detailed analytical discussion of how our testing approach differs from and complements the existing approach in terms of assessing the local continuity (LC) assumption.

Let $X \in \mathcal{X} \subset \mathbb{R}^{d_x}$ be observable covariates. Assuming that all the probability densities in the following equations are well defined, we can write

$$f_{Y_d(r),T_{|r-r_0|}|R,X}(y,t|r,x) = \frac{f_{R|Y_d(r),T_{|r-r_0|},X}(r|y,t,x)}{f_{X|R}(x|r)f_R(r)} f_{Y_d(r),T_{|r-r_0|},X}(y,t,x), \tag{B.1}$$

where $f_{Y_d(r),T_{|r-r_0|}|R,X}(y,t|r,x)$ denotes the conditional density of $(Y_d(r), T_{|r-r_0|})$ given R, X. On the right-hand side of the equation, the continuity of $f_{R|Y_d(r),T_{|r-r_0|},X}(r|y,t,x)$ in r near r_0 is essentially Lee (2008)'s *stronger local continuity* (SLC) assumption (with different notation), which was originally introduced in the sharp RD framework and later discussed in Dong (2018) in the context of the FRD setting. Since the SLC assumption is not directly testable, the existing literature has derived

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tests for two of its implications: (i) the continuity of $f_R(r)$, see for instance McCrary (2008), Otsu, Xu, and Matsushita (2013), Cattaneo, Jansson, and Ma (2020), and Bugni and Canay (2018), and (ii) the continuity of $F_{X|R}(x|r)$ in r, see Canay and Kamat (2018). We can see from equation (B.1) that LC can be considered as an implication of SLC. It is, however, important to note that LC neither implies nor is implied by continuity of $f_R(r)$ and continuity of $F_{X|R}(x|r)$ in r. As a result, there are important empirical scenarios in which the conclusions of the existing tests are not necessarily informative about validity or invalidity of LC, as we illustrate below.

Scenario 1: The existing approach accepts continuity of f_R and $F_{X|R}$ while our approach refutes LC:

This scenario corresponds to the case that the existing approach of testing continuity of f_R and $F_{X|R}$ at $r = r_0$ overlooks the failure of FRD-design, while our approach detects it. To be specific, consider an empirical context in which multiple programs share the same running variable R and the common threshold r_0, e.g., a household can participate in two social programs and both of them use the same poverty index and poverty line to determine eligibility. Let $D, Z \in \{0, 1\}$ denote the treatment statuses, respectively. The researcher is interested in the causal effect of treatment in the first programme, i.e., the effect of D. For simplicity, let us assume that the assignment of the second program is sharp, $Z = \mathbf{1}[R \geq r_0]$. In such a context, the potential outcome model can be written as

$$Y = \underbrace{\{Y_{11}Z + Y_{10}(1-Z)\}}_{Y_1} D + \underbrace{\{Y_{01}Z + Y_{00}(1-Z)\}}_{Y_0}(1-D), \tag{B.2}$$

where $Y_{dz}(r)$, $d \in \{1, 0\}$ and $z \in \{1, 0\}$, are the potential outcomes indexed by the two treatments. As can be seen, if the researcher is unaware of the second treatment, the potential outcome Y_d that she specifies would be $Y_d = Y_{d1}Z + Y_{d0}(1-Z)$. Suppose now $f_{R|Y_{dz}(r),T_{|r-r_0|},X}(r|y,t,x)$ is continuous in r for any y, t, x, then $f_R(r)$ and $F_{X|R}(x|r)$ are continuous. However, the density $f_{R|Y_d(r),T_{|r-r_0|},X}(r|y,t,x)$ can be discontinuous if the second treatment Z affects the outcome. Specifically, since we have (for $d = 1$)

$$\begin{aligned}
&\lim_{r \downarrow r_0} f_{R|Y_{11}Z+Y_{10}(1-Z),T_{|r-r_0|},X}(r|y,t,x) - \lim_{r \uparrow r_0} f_{R|Y_{11}Z+Y_{10}(1-Z),T_{|r-r_0|},X}(r|y,t,x) \\
&\quad = \lim_{r \downarrow r_0} f_{R|Y_{11},T_{|r-r_0|},X}(r|y,t,x) - \lim_{r \uparrow r_0} f_{R|Y_{10},T_{|r-r_0|},X}(r|y,t,x) \\
&\quad = \left[\lim_{r \to r_0} \frac{f_{Y_{11},T_{|r-r_0|}|R,X}(y,t|r,x)}{f_{Y_{11},T_{|r-r_0|}|X}(y,t|x)} - \lim_{r \to r_0} \frac{f_{Y_{10},T_{|r-r_0|}|R,X}(y,t|r,x)}{f_{Y_{10},T_{|r-r_0|}|X}(y,t|x)} \right] f_{R|X}(r_0|x),
\end{aligned}$$

where the first equality follows since the assignment of Z is sharp, and the second equality follows from Bayes rule and continuity of $f_{R|Y_{dz}(r),T_{|r-r_0|},X}(r|y,t,x)$ in r. The two terms in the brackets do not have to cancel out as Y_{10} and Y_{11} are two different potential outcomes, with and without the second treatment. Therefore, in this scenario, learning continuity of $f_R(r)$ and $F_{X|R}(x|r)$ does not inform about failure of LC. Our approach, in contrast, can detect violation of LC, if the distributional differences between Y_{d0} and Y_{d1} at the cut-off leads to violation of the testable implications of Theorem 1 (i) in the main text.

For ease of exposition, we consider binary Z here and interpret it as an unobservable treatment sharing the running variable and cut-off. It is straightforward to generalize the current argument to cases where Z is nonbinary and its discontinuity at r_0 is in terms of its conditional distribution given r. We can also interpret Z as any unobservable factor affecting the outcome, whose distribution changes discontinuously at the cut-off. For instance, the existence of such a Z is often suspected when geographical boundaries are used for regression discontinuity.

Scenario 2: The existing approach rejects continuity of f_R and $F_{X|R}$ while FRD-validity holds (so that our approach does not refute):

This scenario corresponds to the case that FRD-validity holds but the existing approach finds discontinuity of f_R and $F_{X|R}$ at the cut-off. This can happen for a data generating process in which the discontinuity of either $f_R(r)$ or $f_{X|R}(x|r)$ (or both) is compensated exactly by the discontinuity of $f_{R|Y_1(r),T_{|r-r_0|},X}(r|y,t,x)$ in such a way that $f_{Y_d(r),T_{|r-r_0|}|R,X}(y,t|r,x)$ remains continuous. This scenario is not pathological, and is likely to happen in empirical applications where the manipulation is made independently of the potential outcomes. For instance, in the context of the empirical application relating to Maimonides's rule in Israel, Angrist, Lavy, Leder-Luis, and Shany (2019) argues that the presence of discontinuity in the running variable (school enrollment) is mainly due to a school board administration objective to increase their budgets and was "unrelated to socioeconomic characteristics conditional on a few controls" (please refer to Section 5.1 for detailed discussion). This narrative evidence justifies $f_{R|Y_1(r),T_{|r-r_0|},X}(r|y,t,x) = f_{R|X}(r|x)$ in some local neighborhood of the cut-off. This reduces equation (B.1) to

$$f_{Y_1(r),T_{|r-r_0|}|R,X}(y,t|r,x) = \frac{f_{Y_1(r),T_{|r-r_0|},X}(y,t,x)}{f_X(x)}$$

in the local neighborhood of the cut-off, implying $f_{Y_1(r),T_{|r-r_0|}|R,X}(y,t|r,x)$ is continuous at r_0. However, either $f_R(r)$ or $F_{X|R}(x|r)$ (or both) is discontinuous at r_0. This example illustrates that even when the running variable's density is discontinuous, FRD-validity can hold.

Appendix C. Extensions

In this section, we briefly discuss several extensions. First, we discuss the relationship between FRD-validity considered in our paper and other FRD-identifying assumptions considered in the literature. Second, we show how to incorporate conditioning covariates in our test.

C.1. Relationship with other FRD identifying assumptions.

In the LATE framework, de Chaisemartin (2017) argues that the Wald (IV) estimand can have a well-defined causal interpretation under a weaker version of instrument monotonicity. A parallel of his weaker monotonicity condition in the FRD setting can be written as follows: there exists $\epsilon > 0$ such that

$$P(T_{|r-r_0|} = \mathbf{DF}|Y_d(r) = y, R = r) \leq P(T_{|r-r_0|} = \mathbf{C}|Y_d(r) = y, R = r), \quad d \in \{0,1\}, \quad y \in \mathcal{Y}$$

for all $r \in (r_0 - \epsilon, r_0 + \epsilon)$. It can be shown that our Theorem 1 holds by replacing Assumption 1 with this weaker monotonicity assumption and modifying Assumption 2 to include $T = \mathbf{DF}$. That is, inequalities 2 and 3 remain unimprovable testable implications under this weaker version of the local monotonicity assumption.

Bertanha and Imbens (2020) consider an alternative local monotonicity assumption that is more restrictive than Assumption 1.

Assumption C.1 (Strong local monotonicity)**.** *There exists $\epsilon > 0$ such that any individual in the population is classified into one of the following three types based on their treatment selection responses:*

$$T = \begin{cases} \mathbf{A}, & \text{if } D(r) = 1, \text{ for } r \in (r_0 - \epsilon, r_0 + \epsilon), \\ \mathbf{C}, & \text{if } D(r) = 1\{r \geq r_0\}, \text{ for } r \in (r_0 - \epsilon, r_0 + \epsilon), \\ \mathbf{N}, & \text{if } D(r) = 0, \text{ for } r \in (r_0 - \epsilon, r_0 + \epsilon). \end{cases} \tag{C.1}$$

This monotonicity implies that in some neighborhood of the cut-off, compliance status is invariant for any given individual. It can be shown that strengthening Assumption 1 to Assumption C.1 does not yield further testable implications beyond those of Theorem 1 (i), i.e., Theorem 1 (ii) holds true even if Assumption 1 in the main text is replaced by Assumption C.1 above.[1]

The literature has considered the *local independence* assumption,[2] which is a stronger form of identifying assumption than LC.

Assumption C.2 (Local independence)**.** *There exists $\epsilon > 0$ such that for $d = 0, 1$, $(Y_d(r), D(r))$ is jointly independent of R in the neighborhood $(r_0 - \epsilon, r_0 + \epsilon)$ and $\lim_{r \downarrow r_0} Y_d(r) = \lim_{r \uparrow r_0} Y_d(r) \equiv Y_d(r_0)$ a.s.*

The statement of Theorem 1 (i) indeed holds even if LC is replaced by this local independence assumption.

C.2. **Incorporating Covariates.** The standard FRD design does not require covariates to identify treatment effect at the cut-off, but they are often included in practice to increase efficiency. See Imbens and Kalyanaraman (2012), Calonico, Cattaneo, Farrel, and Titiunik (2019), and Hsu and Shen (2019). Another motivation for incorporating the conditioning covariates arises if the potential outcomes can depend on the covariates whose distribution is discontinuous at the cut-off. In this case, RD analysis without conditioning on covariates leads to violation of the local continuity assumption (Frölich and Huber (2019)).

In what follows, we consider testing a version of FRD-validity where local monotonicity and local continuity are imposed conditional on a covariate vector $X \in \mathcal{X} \subset \mathbb{R}^{d_x}$. We allow X to be discrete or continuous. We assume that there are observations near the cut-off point conditioning on each realization x. The conditional version of FRD-validity is stated formally as follows:

Assumption C.3. *The limits $\pi^+(x) \equiv \lim_{r \downarrow r_0} P(D = 1 | R = r, X = x)$ and $\pi^-(x) \equiv \lim_{r \uparrow r_0} P(D = 1 | R = r, X = x)$ exist and $\pi^+(x) \neq \pi^-(x)$ for all $x \in \mathcal{X}$.*

[1] Our proof of Theorem 1 (ii) in Appendix E constructs a distribution of potential outcomes and selection types that, in fact, satisfies Assumption C.1.

[2] This assumption is slightly weaker than the HTV local independence assumption, which involves a local exclusion restriction that rules out causal dependence of Y_d on R in the neighborhood. See Dong and Lewbel (2015), and Dong (2018) for discussion comparing local continuity and HTV local independence, and the restrictions that HTV local independence impose on the distribution of observables.

Assumption C.4 (Local continuity conditional on X)**.** *For $d = 0, 1$, $t \in \{\mathbf{A}, \mathbf{C}, \mathbf{N}\}$, and $B \subseteq \mathcal{Y}$ be a measurable set, the conditional probability $P(Y_d(r) \in B, T_{|r-r_0|} = t | R = r, X = x)$ is continuous in r at r_0, for all $x \in \mathcal{X}$.*

Assumption C.5 (Local monotonicity conditional on X)**.** *Let $t \in \{\mathbf{DF}, \mathbf{I}\}$. There exists a small $\epsilon > 0$ such that $P(T_{|r-r_0|} = t | R = r, X = x) = 0$ for all $r \in (r_0 - \epsilon, r_0 + \epsilon)$ and for all $x \in \mathcal{X}$.*

Theorem 1 can be immediately extended to the conditional version of FRD-validity by conditioning additionally on X. To fit into our testing framework, it is convenient to rewrite the moment inequalities conditional on X in terms of moment inequalities unconditional on X.

To do so, let $Z = (Y, X)$ and $\mathcal{Z}$ be the support of Z. We obtain the following inequalities as the testable implications for Assumptions C.3-C.5: for C a hypercube in $\mathcal{Z}$:

$$\lim_{r \uparrow r_0} \mathbb{E}_P[1\{Z \in C\} D | R = r] - \lim_{r \downarrow r_0} \mathbb{E}_P[1\{Z \in C\} D | R = r] \leq 0, \tag{C.2}$$

$$\lim_{r \downarrow r_0} \mathbb{E}_P[1\{Z \in C\}(1 - D) | R = r] - \lim_{r \uparrow r_0} \mathbb{E}_P[1\{Z \in C\}(1 - D) | R = r] \leq 0. \tag{C.3}$$

In comparison to inequalities (2) and (3), the only difference is that the indicator functions in (C.2) and (C.3) index boxes in $\mathcal{Z}$ instead of the intervals in $\mathcal{Y}$. Accordingly, by defining a class of instrument functions as

$$\begin{aligned}
&\mathcal{G}_z = \{g_\ell(\cdot) = 1(\cdot \in C_\ell) : \ell \equiv (z, c) \in \mathcal{L}\}, \text{ where} \\
&C_\ell = \times_{j=1}^{d_x+1} [z_j, z_j + c] \cap \mathcal{Z} \text{ and} \\
&\mathcal{L} = \left\{ (z, c) : \ c^{-1} = q, \text{ and } q \cdot z_j \in \{0, 1, 2, \cdots, (q-1)\}^{d_x+1} \text{ for } q = 1, 2, \cdots \right\},
\end{aligned} \tag{C.4}$$

we can implement the testing procedure shown in the main text to assess the conditional version of FRD-validity.

C.3. **Joint test.** Our test complements the widely used continuity tests for the distribution of conditioning covariates. Since continuity of the conditional distribution of some covariates at the cut-off is often considered to be supporting evidence for no-selection around the cut-off, it is worthwhile to combine our test with a continuity test.

Suppose we want to test the continuity of the distribution of covariates X at the cut-off *jointly* with the testable implications of (2) and (3). Since continuity of the distribution of X is expressed as a set of local moment *equalities*, we can obtain a joint test by adding the additional set of equality constraints to the null hypothesis.

We hence consider testing the inequalities of Theorem 1 (i) in the main text and the set of equalities indexed by $j \in \mathcal{J}$,

$$v^x(j) \equiv \lim_{r \uparrow r_0} \mathbb{E}_P[1\{X \in C_j^x\}|R = r] - \lim_{r \downarrow r_0} \mathbb{E}_P[1\{X \in C_j^x\}|R = r] = 0.$$

where C_j^x is a hypercube or a quadrant in the space of covariates X, and $\mathcal{J}$ forms a countable collection thereof.

We estimate $v^x(j)$ by $\hat{v}^x(j)$, the difference of two local linear estimators. Following how Andrews and Shi (2013) incorporate moment equalities, we modify the KS test statistic as

$$\hat{S}_n^{joint} = \max \left\{ \sup_{d \in \{0,1\},\ \ell \in \mathcal{L}} \frac{\sqrt{nh} \cdot \hat{v}_{n,d}(\ell)}{\hat{\sigma}_{n,d,\xi}(\ell)}, \quad \sup_{j \in \mathcal{J}} \frac{\sqrt{nh} \cdot |\hat{v}^x(j)|}{\hat{\sigma}^x_{n,\xi}(j)} \right\},$$

where $\hat{\sigma}^x_{n,\xi}(j)$ is an estimator for the asymptotic standard deviation of $\sqrt{nh}(\hat{v}^x(j) - v^x(j))$. Critical values for this test statistic can be obtained by a procedure similar to Algorithm 1 in the main text. Some differences are that for the moment equality constraints, we do not have the moment selection step, and the absolute values are taken for the estimators $\hat{v}^x(j)$ and their bootstrap analogues.

Appendix D. Asymptotic Properties of the Proposed Test

In this appendix, we spell out the regularity conditions and state the theorems that guarantee the asymptotic validity of our test. Their proofs are given in Appendix E.3.

We normalize the support of the observed outcome Y to $[0,1]$.[3] Let $\mathcal{P}$ be the collection of probability distributions of observables (Y, D, R). We denote the Lebesgue density of the running variable, R, by f_R.

Let $h_2(\cdot,\cdot)$ be a covariance kernel on $\mathcal{L} \times \mathcal{L}$. Let $\mathcal{H}_2$ be the collection of all possible covariance kernel functions on $\mathcal{L} \times \mathcal{L}$. For any pair of $h_2^{(1)} \in \mathcal{H}_2$ and $h_2^{(2)} \in \mathcal{H}_2$, we define the distance between them as

$$d(h_2^{(1)}, h_2^{(2)}) = \sup_{\{\ell_1, \ell_2 \in \mathcal{L}\}} |h_2^{(1)}(\ell_1, \ell_2) - h_2^{(2)}(\ell_1, \ell_2)|. \tag{D.1}$$

[3]This support normalization is without loss of generality. If not, we can define $\widetilde{Y} = \Phi(Y)$ where $\Phi(\cdot)$ is the CDF of standard normal, as in the first step of Algorithm 1.

Let $\sigma_{P,d}(\ell_1,\ell_2|r) = Cov_P(g_{\ell_1}(Y)D^d(1-D)^{1-d}, g_{\ell_2}(Y)D^d(1-D)^{1-d}|R=r)$ for $d=1,0$. We denote their right and left limits at r_0 by $\sigma_{P,d,+}(\ell_1,\ell_2) = \lim_{r\downarrow r_0}\sigma_{P,d}(\ell_1,\ell_2|r)$ and $\sigma_{P,d,-}(\ell_1,\ell_2) = \lim_{r\uparrow r_0}\sigma_{P,d}(\ell_1,\ell_2|r)$, respectively. Existence of these limits is implied by the set of assumptions in Assumption D.1, below.

For $j = 0,1,2,\ldots$, let $\vartheta_j = \int_0^\infty u^j K(u)du$. Let $f_R^+(r_0) = \lim_{r\downarrow r_0} f_R(r_0)$ and $f_R^-(r_0) = \lim_{r\uparrow r_0} f_R(r_0)$. For $d=0,1$ and $\star = +,-$, define

$$h_{2,P,d,\star}(\ell_1,\ell_2) = \frac{\int_0^\infty(\vartheta_2 - u\vartheta_1)^2K^2(u)du\cdot\sigma_{P,d,\star}(\ell_1,\ell_2)}{c_\star f_R^\star(r_0)(\vartheta_2\vartheta_0-\vartheta_1^2)^2}, \tag{D.2}$$

which is the covariance kernel of the limiting process of $\sqrt{nh}(\hat{m}_{d,\star}(\ell) - m_{P,d,\star}(\ell))$, with undersmoothing bandwidths. It can be shown that the covariance kernel of the limiting processes of $\sqrt{nh}(\hat{v}_d(\ell) - v_{P,d}(\ell))$ is $h_{2,P,d} = h_{2,P,d,+} + h_{2,P,d,-}$.

We denote the v-th derivative of $m_{P,d}(\ell,r) = \mathbb{E}_P[g_\ell(Y)D^d(1-D)^{1-d}|R=r]$ with respect to the running variable by $m_{P,d}^{(v)}(\ell,r)$, $d=1,0$. The v-th derivative of f_R is denoted similarly. For $\delta>0$, define $\mathcal{N}_\delta(r_0) = \{r:\ |r-r_0|<\delta\}$ as a neighborhood of r around r_0. Let $\mathcal{N}_\delta^+(r_0) = \{r:\ 0<r-r_0<\delta\}$ and $\mathcal{N}_\delta^-(r_0) = \{r:\ 0<r_0-r<\delta\}$ be one-sided open neighborhoods excluding r_0.

Assumption D.1. *Let f_R be common for all $P\in\mathcal{P}$. There exist $\delta>0$, $\epsilon>0$, $0<\bar{f}_R<\infty$, and $0\le M<\infty$ such that for all $P\in\mathcal{P}$,*

(i) $f_R(r)>\epsilon$ on $\mathcal{N}_\delta(r_0)$.

(ii) $f_R(r)$ is continuous and bounded from above by $\bar{f}_R$ on $\mathcal{N}_\delta^+(r_0)\cup\mathcal{N}_\delta^-(r_0)$, and $f_R^+(r_0)$ and $f_R^-(r_0)$ exist.

(iii) $f_R(r)$ is twice continuously differentiable in r on $\mathcal{N}_\delta^+(r_0)\cup\mathcal{N}_\delta^-(r_0)$ and $|f_R^{(1)}(r)|\le M$ and $|f_R^{(2)}(r)|\le M$ on $\mathcal{N}_\delta^+(r_0)\cup\mathcal{N}_\delta^-(r_0)$;

(iv) for $d=0,1$ and for all $\ell\in\mathcal{L}$, $m_{P,d}(\ell,r)$ is twice continuously differentiable in r on $\mathcal{N}_\delta^+(r_0)\cup\mathcal{N}_\delta^-(r_0)$;

(v) for $d=0,1$ and for all $\ell\in\mathcal{L}$, $|m_{P,d}^{(1)}(\ell,r)|\le M$ and $|m_{P,d}^{(2)}(\ell,r)|\le M$ on $\mathcal{N}_\delta^+(r_0)\cup\mathcal{N}_\delta^-(r_0)$;

Assumption D.1 (*iii*)-(*v*) imply that with undersmoothing bandwidths, the bias terms of $\hat{v}_1(\ell)$ and $\hat{v}_0(\ell)$ are asymptotically negligible uniformly over $\ell\in\mathcal{L}$. Note that Assumption D.1 does not restrict the support of Y and allows Y to be discrete, continuous, or some mixture of the two. Note also that we allow $f_R(r)$ to be discontinuous at the cut-off, reflecting the fact that the testable implications of FRD-validity that we are focusing on do not require continuity of $f_R(r)$ at the cut-off.

Assumption D.2. *The kernel function $K(\cdot)$ and bandwidth h satisfy*

(i) $K(\cdot)$ *is nonnegative, symmetric, bounded by* $\bar{K} < \infty$*, and has a compact support (say* $[-1, 1]$*),*

(ii) $\int_{\mathbb{R}} K(u)du = 1$*, and* $\int_{\mathbb{R}} u^2 K(u)du > 0$*,*

(iii) $h \to 0$*,* $nh \to \infty$ *and* $nh^5 \to 0$ *as* $n \to \infty$*.*

Assumption D.2 is standard, and the triangular kernel used in our Monte Carlo studies and empirical applications satisfies this assumption. Note that $nh^5 \to 0$ as $n \to \infty$ corresponds to an undersmoothing choice of bandwidth so that the bias term of $\hat{v}_{n,d}$ converges to zero even after $\sqrt{nh}$ is multiplied.

Assumption D.3. *Let* $\{U_i : 1 \leq i \leq n\}$ *be a sequence of i.i.d. random variables* $E[U] = 0$*,* $E[U^2] = 1$*, and* $E[|U|^4] < M_1$ *for some* $M_1 < \infty$*, and* $\{U_i : 1 \leq i \leq n\}$ *is independent of the sample* $\{(Y_i, D_i, R_i) : 1 \leq i \leq n\}$*.*

Assumption D.3 is standard for the multiplier bootstrap (see, e.g., Hsu (2016)). Note the standard normal distribution for U satisfies Assumption D.3.

Assumption D.4. $\{a_n\}$ *is a sequence of nonnegative numbers satisfying* $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty} a_n/\sqrt{nh} = 0$*.* $\{B_n\}$ *is a sequence of nonnegative numbers that is nondecreasing,* $\lim_{n\to\infty} B_n = \infty$ *and* $\lim_{n\to\infty} B_n/a_n = 0$*.*

In our Monte Carlo study and empirical applications, we specify $a_n = (0.3\ln(n))^{1/2}$ and $B_n = (0.4\ln(n)/\ln\ln(n))^{1/2}$ following Andrews and Shi (2013, 2014).

Let $\mathcal{P}_0 \subset \mathcal{P}$ be the set of distributions of observables that satisfy the null hypothesis given in equation (5) in the main text. The next assumption states that $\mathcal{P}_0$ contains a distribution of data that satisfies the moment inequalities $\{v_{P,d}(\ell) : d = 0, 1, \ell \in \mathcal{L}\}$ with equality for some $\ell \in \mathcal{L}$.

Assumption D.5. *Let* $\mathcal{L}^o_{P,d} \equiv \{\ell \in \mathcal{L} : v_{P,d}(\ell) = 0\}$*. There exists* $P_c \in \mathcal{P}_0$ *such that* *(i) Either $\mathcal{L}^o_{P_c,1}$ or $\mathcal{L}^o_{P_c,0}$ under P_c is nonempty.*

(ii) For $d = 0, 1$, $h_{2,P_c,d,+} \in \mathcal{H}_{2,cpt}$ and $h_{2,P_c,d,-} \in \mathcal{H}_{2,cpt}$, where $\mathcal{H}_{2,cpt}$ is a compact subset of $\mathcal{H}_2$ with respect to the norm defined in equation (D.1).

(iii) Either $h_{2,P_c,1} = h_{2,P_c,1,+} + h_{2,P_c,1,-}$ restricted to $\mathcal{L}^o_{P_c,1} \times \mathcal{L}^o_{P_c,1}$ is not a zero function or $h_{2,P_c,0} = h_{2,P_c,0,+} + h_{2,P_c,0,-}$ restricted to $\mathcal{L}^o_{P_c,0} \times \mathcal{L}^o_{P_c,0}$ is not a zero function.

Theorem D.1. *Suppose Assumptions D.1-D.4 hold. Then, for every compact subset $\mathcal{H}_{2,cpt}$ of $\mathcal{H}_2$, the following claims hold for the testing procedure presented in Algorithm 1:*

(a) $\limsup_{n\to\infty} \sup_{\{P\in\mathcal{P}_0:\ d\in\{0,1\}, h_{2,P,d,+}, h_{2,P,d,-}\in\mathcal{H}_{2,cpt}\}} P(\widehat{S}_n > \hat{c}_\eta(\alpha)) \leq \alpha.$

(b) If Assumption D.5 also holds, then

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \sup_{\{P\in\mathcal{P}_0:\ d\in\{0,1\}, h_{2,P,d,+}, h_{2,P,d,-}\in\mathcal{H}_{2,cpt}\}} P(\widehat{S}_n > \hat{c}_\eta(\alpha)) = \alpha.$$

Theorem D.1 (a) shows that our test has asymptotically uniformly correct size over a compact set of covariance kernels. Theorem D.1 (b) shows that our test is at most infinitesimally conservative asymptotically when the null contains at least one P_c defined in Assumption D.5. Theorem D.1 extends Theorem 2 of Andrews and Shi (2013) and Theorem 5.1 of Hsu (2017) to local moment inequalities in the context of RD designs.

The next theorem shows consistency of our test against a fixed alternative.

Theorem D.2. *Suppose Assumptions D.1-D.4 hold and $\alpha < 1/2$. If there exists $\ell \in \mathcal{L}$ such that either $\nu_{P_1,1}(\ell) > 0$ or $\nu_{P_1,0}(\ell) > 0$, then $\lim_{n\to\infty} P(\widehat{S}_n > \hat{c}_\eta(\alpha)) = 1$.*

We can also show that our test is unbiased against some $\sqrt{nh}$-local alternatives. We consider a sequence of $P_n \in \mathcal{P}\backslash\mathcal{P}_0$ such that

$$\nu_{P_n,d}(\ell) = \nu_{P_c,d}(\ell) + \frac{\delta_d(\ell)}{\sqrt{nh}}, \tag{D.3}$$

for $d = 1, 0$ and some $P_c \in \mathcal{P}_0$ defined in Assumption D.5. Here, $\delta_d(\ell) > 0$ specifies local violation of the null hypothesis inside the interval $\ell \in \mathcal{L}$. We consider local alternatives that satisfy the next set of assumptions:

Assumption D.6. *A sequence of local alternatives $\{P_n \in \mathcal{P}\backslash\mathcal{P}_0 :\ n \geq 1\}$ satisfies the following conditions:*

(i) ([D.3](#)) *holds under* P_n,

(ii) *for* $d = 0, 1$, $\delta_d(\ell) \geq 0$ *if* $\ell \in \mathcal{L}^o_{P_c,d}$,

(iii) *for* $d = 0, 1$, $\delta_d(\ell) > 0$ *for some* $\ell \in \mathcal{L}^o_{P_c,d}$.

(iv) *for* $d = 0, 1$, $\lim_{n\to\infty} d(h_{2,P_n,d,+}, h^*_{2,d,+}) = 0$ *and* $\lim_{n\to\infty} d(h_{2,P_n,d,-}, h^*_{2,d,-}) = 0$ *for some* $h^*_{2,d,+} \in \mathcal{H}_2$ *and* $h^*_{2,d,-} \in \mathcal{H}_2$.

Assumption D.6 (i) requires that the local alternatives converge to a boundary null P_c at rate $(nh)^{-1/2}$. Assumption D.6 (ii) ensures that our test is unbiased and Assumption D.6 (iii) makes sure that each P_n in the sequence is not in $\mathcal{P}_0$. Assumption D.6 (iv) restricts the asymptotic behavior of the covariance kernels as considered in LA1(c) of Andrews and Shi (2013).

The following theorem shows that the asymptotic local power of our test is greater than or equal to α when η tends to zero, i.e., our test is unbiased against those local alternatives that satisfy Assumption D.6.

Theorem D.3. *Suppose Assumptions D.1 to D.4 hold and* $\alpha < 1/2$. *If a sequence of local alternatives* $\{P_n : n \geq 1\}$ *satisfies Assumption D.6, then* $\lim_{\eta\to 0} \lim_{n\to\infty} P(\widehat{S}_n > \hat{c}_\eta(\alpha)) \geq \alpha$.

Appendix E. Proofs

We first introduce a lemma that allows us to extend inequalities (2) and (3) to any Borel set in $\mathcal{Y}$.

Lemma E.1. *Under the conditions of Theorem 1 (i), inequalities (2) and (3) hold for any closed interval* $[y', y]$, $-\infty \leq y' \leq y \leq \infty$, *if and only if they hold for any Borel set in* $\mathcal{Y}$.

Proof. We focus on inequality (2). The claim concerning inequality (3) can be shown analogously. The "if" part is trivial. We apply Andrews and Shi (2013, Lemma C1) to show the "only if" part. Let $\mathcal{C} \equiv \{[y, y'] : -\infty \leq y \leq y' \leq \infty\}$ be the class of intervals and C be a generic element of $\mathcal{C}$. Let $\mu_1(\cdot) = \lim_{r\downarrow r_0} \mathbb{E}_P[1\{Y \in \cdot\}D|R = r] - \lim_{r\uparrow r_0} \mathbb{E}_P[1\{Y \in \cdot\}D|R = r]$, which is a well-defined set function if Assumptions 1 and 2 hold. See the proof of Theorem 1 (i) below for existence of the left and right limits of $E_P[1\{Y \in \cdot\}D|R = r]$. It then holds that $\mu_1 : \mathcal{C} \to \mathbb{R}$ is a bounded and countably additive set function satisfying $\mu_1(\emptyset) = 0$ and $\mu_1(C) \geq 0$ for any C. Applying Andrews and Shi (2013, Lemma C1), since the smallest σ-algebra generated by $\mathcal{C}$ coincides with the Borel σ-algebra $\mathcal{B}(\mathcal{Y})$, it follows that $\mu_1(C_\ell) \geq 0$ for any $C_\ell \in \mathcal{L}$ implies that $\mu_1(B) \geq 0$ for any $B \in \mathcal{B}(\mathcal{Y})$. □

E.1. **Proof of Theorem 1: Claim (i):** Let $B \subset \mathbb{R}$ be an arbitrary closed interval. We have

$$\lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y \in B\}D|R = r_0+\epsilon] \geq \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} \in \{\mathbf{A},\mathbf{C}\}\}|R = r_0+\epsilon]$$
$$= \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} = \mathbf{C}\}|R = r_0+\epsilon] + \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} = \mathbf{A}\}|R = r_0+\epsilon],$$

where the first inequality follows since the set of selection types such that $D(r) = 1$ at $r_0 \leq r < r_0+\epsilon$ includes $\{\mathbf{A},\mathbf{C}\}$. On the other hand, we have

$$\lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y \in B\}D|R = r_0+\epsilon]$$
$$\leq \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} \in \{\mathbf{A},\mathbf{C}\}\}|R = r_0+\epsilon] + \lim_{\epsilon\to 0}P(T_{|r-r_0|} = \mathbf{I}|R = r_0+\epsilon)$$
$$= \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} = \mathbf{C}\}|R = r_0+\epsilon] + \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} = \mathbf{A}\}|R = r_0+\epsilon]$$

where the third line follows by Assumption 1. Hence,

$$\lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y \in B\}D|R = r_0+\epsilon]$$
$$= \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} = \mathbf{C}\}|R = r_0+\epsilon] + \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} = \mathbf{A}\}|R = r_0+\epsilon]. \tag{E.1}$$

Similarly, we have $\lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y \in B\}D|R = r_0-\epsilon] \geq \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} = \mathbf{A}\}|R = r_0-\epsilon]$ and

$$\lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y \in B\}D|R = r_0-\epsilon]$$
$$\leq \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} = \mathbf{A}\}|R = r_0-\epsilon] + \lim_{\epsilon\to 0}P(T_{|r-r_0|} \in \{\mathbf{I},\mathbf{DF}\}|R = r_0-\epsilon)$$
$$= \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} = \mathbf{A}\}|R = r_0-\epsilon],$$

implying

$$\lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y \in B\}D|R = r_0-\epsilon] = \lim_{\epsilon\to 0}\mathbb{E}_P[1\{Y_1(r) \in B, T_{|r-r_0|} = \mathbf{A}\}|R = r_0-\epsilon]. \tag{E.2}$$

Taking the difference of equation (E.1) and equation (E.2) and employing Assumption 2 leads to the desired inequality:

$$\begin{aligned}
&\lim_{r\downarrow r_0}\mathbb{E}_P[1\{Y\in B\}D|R=r]-\lim_{r\uparrow r_0}\mathbb{E}_P[1\{Y\in B\}D|R=r]\\
=&\lim_{r\downarrow r_0}\mathbb{E}_P[1\{Y_1(r)\in B, T_{|r-r_0|}\in\{\mathbf{C}\}\}|R=r]\geq 0.
\end{aligned}\tag{E.3}$$

Similarly we can show that

$$\begin{aligned}
&\lim_{r\uparrow r_0}\mathbb{E}_P[1\{Y\in B\}(1-D)|R=r]-\lim_{r\downarrow r_0}\mathbb{E}_P[1\{Y\in B\}(1-D)|R=r]\\
=&\lim_{r\downarrow r_0}\mathbb{E}_P[1\{Y_0(r)\in B, T_{|r-r_0|}\in\{\mathbf{C}\}\}|R=r]\geq 0.
\end{aligned}\tag{E.4}$$

Note that the proof is also valid when Assumption 1 is replaced by Assumption C.1.

Claim (ii): Suppose that the distribution of observables (Y,D,R) satisfies inequalities (2) and (3). By Lemma E.1, they hold for an arbitrary Borel set. By the absolute continuity assumption, we have the conditional density of (Y,D) given R denoted by $f_{Y,D|R}(y,d|r)$. We denote the left and right limits of $f_{Y,D|R}$ at r_0 by $f_{Y,D|R}(y,d|r_{0,-})=\lim_{r\uparrow r_0}f_{Y,D|R}(y,d|r)$ and $f_{Y,D|R}(y,d|r_{0,+})=\lim_{r\downarrow r_0}f_{Y,D|R}(y,d|r)$, respectively.

In what follows, we construct a joint distribution of potential variables $(\tilde{Y}_1(r),\tilde{Y}_0(r),\tilde{D}(r):r\in\mathcal{R})$ that satisfies Assumptions 1 and 2 and matches with the given distribution of observables.

First, for $d\in\{0,1\}$, consider outcome responses that are invariant to the running variable, $\tilde{Y}_d(r)=\tilde{Y}_d(r')$ for all $r,r'\in\mathcal{R}$, a.s., i.e., the running variable has no direct causal impact for anyone in the population. We can hence drop index r from the notation of the potential outcomes and reduce them to $(\tilde{Y}_1,\tilde{Y}_0)\in\mathcal{Y}^2$. For the treatment selection response to the running variable, consider that only the following selection responses are allowed in the population:

$$\tilde{D}(r)=\begin{cases}1\{r\geq r_0\}, & \text{labeled as } \tilde{T}=\mathbf{C}\\ 1, & \text{labeled as } \tilde{T}=\mathbf{A}\\ 0, & \text{labeled as } \tilde{T}=\mathbf{N}.\end{cases}$$

With these simplifications, we construct a joint distribution of $(\tilde{Y}_1(r),\tilde{Y}_0(r),\tilde{D}(r):r\in\mathcal{R})$ given R by constructing a joint distribution of $(\tilde{Y}_1,\tilde{Y}_0,\tilde{T})\in\mathcal{Y}^2\times\{\mathbf{C},\mathbf{A},\mathbf{N}\}$ given R, where $\tilde{T}$ does not vary in $|r-r_0|$. To distinguish the probability law of observables corresponding to the given sampling process and the probability law of $(\tilde{Y}_1, \tilde{Y}_0, \tilde{T})$ to be constructed, we use P and f to denote the former probability law and its density, and $\mathbb{P}$ to denote the latter probability law.

Let $B \subset \mathbb{R}$ be an arbitrary Borel Set. For the always-takers' potential outcome distributions, consider

$$\mathbb{P}(\tilde{Y}_1 \in B, \tilde{T} = \mathbf{A}|r) = \begin{cases} P(Y \in B, D = 1|r), & \text{for } r < r_0, \\ \int_B \min \left\{ \begin{matrix} f_{Y,D|R}(y, D = 1|r_{0,-}), \\ f_{Y,D|R}(y, D = 1|r) \end{matrix} \right\} d\mu, & \text{for } r \geq r_0, \end{cases}$$

and

$$\mathbb{P}(\tilde{Y}_0 \in B, \tilde{T} = \mathbf{A}|r) = \begin{cases} Q(B)P(D = 1|r), & \text{for } r < r_0, \\ Q(B) \int_{\mathcal{Y}} \min \left\{ \begin{matrix} f_{Y,D|R}(y, D = 1|r_{0,-}), \\ f_{Y,D|R}(y, D = 1|r) \end{matrix} \right\} d\mu, & \text{for } r \geq r_0, \end{cases},$$

where $Q(\cdot)$ is an arbitrary probability measure on $\mathcal{Y}$. The joint distribution of $(\tilde{Y}_1, \tilde{Y}_0, \tilde{T} = \mathbf{A})$ can be constructed by coupling these distributions assuming, for instance, that $\tilde{Y}_1$ and $\tilde{Y}_0$ are independent conditional on $(\tilde{T}, R)$.

For the never-takers' potential outcome distributions, consider

$$\mathbb{P}(\tilde{Y}_0 \in B, \tilde{T} = \mathbf{N}|r) = \begin{cases} P(Y \in B, D = 0|r), & \text{for } r \geq r_0, \\ \int_B \min \left\{ \begin{matrix} f_{Y,D|R}(y, D = 0|r_{0,+}), \\ f_{Y,D|R}(y, D = 0|r) \end{matrix} \right\} d\mu, & \text{for } r < r_0, \end{cases}$$

and

$$\mathbb{P}(\tilde{Y}_1 \in B, \tilde{T} = \mathbf{N}|r) = \begin{cases} Q(B)P(D = 0|r), & \text{for } r \geq r_0, \\ Q(B) \int_{\mathcal{Y}} \min \left\{ \begin{matrix} f_{Y,D|R}(y, D = 0|r_{0,+}) \\ f_{Y,D|R}(y, D = 0|r) \end{matrix} \right\} d\mu, & \text{for } r < r_0. \end{cases}$$

For the compliers' potential outcome distributions, if $\pi^+ = \pi^-$, we specify that no compliers exist in the population. If $\pi^+ > \pi^-$, consider

$$
\begin{aligned}
&\mathbb{P}(\tilde{Y}_1 \in B, \tilde{T} = \mathbf{C}|r) \\
&= \begin{cases}
P(Y \in B, D = 1|r) - \int_B \min \begin{Bmatrix} f_{Y,D|R}(y, D = 1|r_{0,-}), \\ f_{Y,D|R}(y, D = 1|r) \end{Bmatrix} d\mu, & \text{for } r \geq r_0, \\
(\pi^+ - \pi^-)^{-1} \left[P(D = 1|r) - \int_{\mathcal{Y}} \min \begin{Bmatrix} f_{Y,D|R}(y, D = 1|r_{0,-}), \\ f_{Y,D|R}(y, D = 1|r) \end{Bmatrix} d\mu \right] & \\
\times \left[\lim_{r \downarrow r_0} P(Y \in B, D = 1|r) - \lim_{r \uparrow r_0} P(Y \in B, D = 1|r) \right], & \text{for } r < r_0.
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{P}(\tilde{Y}_0 \in B, \tilde{T} = \mathbf{C}|r) \\
&= \begin{cases}
P(Y \in B, D = 0|r) - \int_B \min \begin{Bmatrix} f_{Y,D|R}(y, D = 0|r_{0,+}), \\ f_{Y,D|R}(y, D = 0|r) \end{Bmatrix} d\mu, & \text{for } r < r_0, \\
(\pi^+ - \pi^-)^{-1} \left[P(D = 1|r) + - \int_{\mathcal{Y}} \min \begin{Bmatrix} f_{Y,D|R}(y, D = 1|r_{0,-}), \\ f_{Y,D|R}(y, D = 1|r) \end{Bmatrix} d\mu \right] & \\
\times \left[\lim_{r \uparrow r_0} P(Y \in B, D = 0|r) - \lim_{r \downarrow r_0} P(Y \in B, D = 0|r) \right], & \text{for } r \geq r_0.
\end{cases}
\end{aligned}
$$

If the distribution of (Y, D, R) satisfies the testable implications shown in the first claim, then it can be shown that the conditional distribution of $(\tilde{Y}_1, \tilde{Y}_0, \tilde{T})$ given $R = r$ constructed in this way is a proper probability distribution (i.e., nonnegative, additive, and sum up to one) for all r. We can also confirm that the constructed distribution of $(\tilde{Y}_1, \tilde{Y}_0, \tilde{T})$ given R matches with the distribution of observables, i.e., it satisfies, for any $d = 1, 0$, $r \in \mathcal{R}$, and measurable set $B \subset \mathcal{Y}$,

$$P(Y \in B, D = d|r) = \sum_{\tilde{T}: \tilde{D}(r) = d} \mathbb{P}(\tilde{Y}_d \in B, \tilde{T}|r).$$

We now check the conditional distribution of $(\tilde{Y}_1, \tilde{Y}_0, \tilde{T})$ given R constructed above satisfies Assumptions 1 and 2. First, by the construction of treatment selection response, $\mathbb{P}(\tilde{T} = \{\mathbf{DF}, \mathbf{I}\}|r) = 0$ for any r. Hence, Assumption 1 holds.

To check Assumption 2, note that

$$\lim_{r\downarrow r_0}\mathbb{P}(\tilde{Y}_1\in B,\tilde{T}=\mathbf{A}|r)=\lim_{r\downarrow r_0}\int_B \min\left\{\begin{array}{c} f_{Y,D|R}(y,D=1|r_{0,-}),\\ f_{Y,D|R}(y,D=1|r)\end{array}\right\}d\mu$$

$$=\int_B \min\left\{\begin{array}{c} f_{Y,D|R}(y,D=1|r_{0,-}),\\ f_{Y,D|R}(y,D=1|r_{0,+})\end{array}\right\}d\mu=\int_B f_{Y,D|R}(y,D=1|r_{0,-})d\mu$$

$$=\lim_{r\uparrow r_0}P(Y\in B,D=1|r)=\lim_{r\uparrow r_0}\mathbb{P}(\tilde{Y}_1\in B,\tilde{T}=\mathbf{A}|r),$$

where the third equality follows by the assumption that the distribution of (Y,D,R) satisfies inequality (2). Hence, $\mathbb{P}(\tilde{Y}_1,\tilde{T}=\mathbf{A}|r)$ is continuous at $r=r_0$. Similar arguments apply to show that $\mathbb{P}(\tilde{Y}_0,\tilde{T}=\mathbf{A}|r)$, $\mathbb{P}(\tilde{Y}_1,\tilde{T}=\mathbf{N}|r)$, and $\mathbb{P}(\tilde{Y}_0,\tilde{T}=\mathbf{N}|r)$ are all continuous at r_0. For compliers, we have

$$\lim_{r\downarrow r_0}\mathbb{P}(\tilde{Y}_1\in B,\tilde{T}=\mathbf{C}|r)$$

$$=\lim_{r\downarrow r_0}\left[P(Y\in B,D=1|r)-\int_B \min\left\{\begin{array}{c} f_{Y,D|R}(y,D=1|r_{0,-}),\\ f_{Y,D|R}(y,D=1|r)\end{array}\right\}d\mu\right]$$

$$=\lim_{r\downarrow r_0}P(Y\in B,D=1|r)-\lim_{r\uparrow r_0}P(Y\in B,D=1|r).$$

Also, by noting $\lim_{r\downarrow r_0}\int_{\mathcal{Y}}\min\left\{\begin{array}{c} f_{Y,D|R}(y,D=1|r_{0,-}),\\ f_{Y,D|R}(y,D=1|r)\end{array}\right\}d\mu=\pi^-$, we obtain

$$\lim_{r\uparrow r_0}\mathbb{P}(\tilde{Y}_1\in B,\tilde{T}=\mathbf{C}|r)=\lim_{r\downarrow r_0}P(Y\in B,D=1|r)-\lim_{r\uparrow r_0}P(Y\in B,D=1|r).$$

Hence, we have shown that the constructed distribution of $(\tilde{Y}_1,\tilde{Y}_0,\tilde{T})$ given R satisfies Assumption 2. This completes the proof of the second claim.

E.2. Identification of the compliers' potential outcome distributions.

Proposition E.1. *If Assumptions 1 to 3 hold, then the compliers' potential outcome distributions at the cut-off,*

$$\begin{aligned} F_{Y_1(r_0)|\mathbf{C},R=r_0}(y) &\equiv \lim_{r\to r_0}P\left(Y_1(r)\le y|T_{|r-r_0|}=\mathbf{C},R=r\right),\\ F_{Y_0(r_0)|\mathbf{C},R=r_0}(y) &\equiv \lim_{r\to r_0}P\left(Y_0(r)\le y|T_{|r-r_0|}=\mathbf{C},R=r\right),\end{aligned}$$

are identified by

$$
\begin{aligned}
F_{Y_1(r_0)|\mathbf{C},R=r_0}(y) &= \frac{\lim\limits_{r\downarrow r_0}\mathbb{E}_P[1\{Y \leq y\}D|R=r] - \lim\limits_{r\uparrow r_0}\mathbb{E}_P[1\{Y \leq y\}D|R=r]}{\pi^+ - \pi^-}, \\
F_{Y_0(r_0)|\mathbf{C},R=r_0}(y) &= \frac{\lim\limits_{r\uparrow r_0}\mathbb{E}_P[1\{Y \leq y\}(1-D)|R=r] - \lim\limits_{r\downarrow r_0}\mathbb{E}_P[1\{Y \leq y\}(1-D)R=r]}{\pi^+ - \pi^-}.
\end{aligned}
$$

Proof. We first note that under Assumptions 1 and 2, $\pi^+ - \pi^- = \lim_{r\to r_0} P(T_{|r-r_0|} = \mathbf{C}|\mathbf{R} = \mathbf{r})$. Based on (E.3) in the proof of Theorem 1, we have

$$
\begin{aligned}
& \lim_{r\downarrow r_0}\mathbb{E}_P[1\{Y \leq y\}D|R=r] - \lim_{r\uparrow r_0}\mathbb{E}_P[1\{Y \leq y\}D|R=r] \\
= \;& F_{Y_1(r_0)|\mathbf{C},R=r_0}(y) \cdot \lim_{r\to r_0} P(T_{|r-r_0|} = \mathbf{C}|\mathbf{R} = \mathbf{r}) \\
= \;& F_{Y_1(r_0)|\mathbf{C},R=r_0}(y) \cdot (\pi^+ - \pi^-)
\end{aligned}
$$

Hence, the identification result of $F_{Y_1(r_0)|\mathbf{C},R=r_0}(y)$ is shown.

The identification result for $F_{Y_0(r_0)|\mathbf{C},R=r_0}(y)$ can be shown similarly by using equation (E.4). We omit the details for brevity. □

E.3. **Lemmas and Proofs for Theorems in Appendix D.** We show three lemmas that lead to the theorems in Appendix D.

We first present a lemma that shows a Bahadur representation for $\hat{m}_{d,\star}$, $d = 0,1$ and $\star = +,-$, uniform in $\ell \in \mathcal{L}$ and $P \in \mathcal{P}$ subject to Assumption D.1. This lemma extends the undersmoothing case of Lemma 1 in Chiang, Hsu, and Sasaki (2017) by providing an approximation that is also uniform over the data generating processes $P \in \mathcal{P}$. It also modifies the undersmoothing case of Theorem 1 in Lee, Song, and Whang (2015) by focusing on the boundary point and uniformity over the class of intervals rather than quantiles.

Given a class of data generating processes $\mathcal{P}$, we say that a sequence of random variables $\{Z_n\}$ converges in probability to zero $\mathcal{P}$-uniformly if $\sup_{\{P\in\mathcal{P}\}} P(|Z_n| > \epsilon) \to 0$ as $n \to \infty$ for any $\epsilon > 0$, which we denote by $Z_n = o_{\mathcal{P}}(1)$.

Lemma E.2. *Let $\mathcal{P}$ be a class of data generating processes satisfying Assumption D.1, and $\hat{m}_{d,\star}$, $m_{P,d}$, and $m_{P,d,\star}$, $d = 1, 0$ and $\star = +, -$, be as defined in Appendix A. Under Assumption D.2,*

$$\sup_{\{\ell \in \mathcal{L}\}} \left| \sqrt{nh}(\hat{m}_{d,\star}(\ell) - m_{P,d,\star}(\ell)) - \frac{1}{\sqrt{nh}} \sum_{i=1}^{n} w_i^{\star} \mathcal{E}_{d,i}(\ell) \right| = o_{\mathcal{P}}(1), \tag{E.5}$$

where

$$\begin{aligned} w_i^+ &= \frac{\left[\vartheta_2 - \vartheta_1 \left(\frac{R_i - r_0}{h_+}\right)\right] K\left(\frac{R_i - r_0}{h_+}\right) 1\{R_i \geq r_0\}}{c_+ f_R^+(r_0)(\vartheta_0 \vartheta_2 - \vartheta_1^2)}, \\ w_i^- &= \frac{\left[\vartheta_2 + \vartheta_1 \left(\frac{R_i - r_0}{h_-}\right)\right] K\left(\frac{R_i - r_0}{h_-}\right) 1\{R_i < r_0\}}{c_- f_R^-(r_0)(\vartheta_0 \vartheta_2 - \vartheta_1^2)}, \\ \mathcal{E}_{d,i}(\ell) &= g_\ell(Y_i) D_i^d (1 - D_i)^{1-d} - m_{P,d}(\ell, R_i). \end{aligned}$$

Proof. We provide a proof for the case of $d = 1$ and $\star = +$ only, as the proofs for the other cases are similar. Substituting the mean value expansion, $g_\ell(Y_i) D_i = m_{P,1}(\ell, R_i) + \mathcal{E}_{1,i}(\ell) = m_{P,1,+}(\ell) + h_+ m_{P,1}^{(1)}(\ell, r_0) \left(\frac{R_i - r_0}{h_+}\right) + \frac{h_+^2}{2} m_{P,1}^{(2)}(\ell, \tilde{R}_i) \left(\frac{R_i - r_0}{h_+}\right)^2 + \mathcal{E}_{1,i}(\ell)$, $\tilde{R}_i \in [0, R_i]$, we obtain

$$\begin{aligned} &\sqrt{nh}\left[\hat{m}_{1,+}(\ell) - m_{P,1,+}(\ell)\right] \\ =&\sqrt{nh^3} \cdot c_+ \sum_{i=1}^{n} w_{n,i}^+ m_{P,1}^{(1)}(\ell, r_0) \left(\frac{R_i - r_0}{h_+}\right) + \sqrt{nh^5} \cdot \frac{c_+^2}{2} \sum_{i=1}^{n} w_{n,i}^+ m_{P,1}^{(2)}(\ell, \tilde{R}_i) \left(\frac{R_i - r_0}{h_+}\right)^2 \qquad \text{(E.6)} \\ &+ \sqrt{nh} \cdot \sum_{i=1}^{n} w_{n,i}^+ \mathcal{E}_{1,i}(\ell) \qquad \text{(E.7)} \end{aligned}$$

The first order conditions for the local linear regression implies the first term in (E.6) is zero. By the boundedness of $m_{P,1}^{(2)}$ (Assumption D.1) (iv), the absolute value of the second term in (E.6) can be bounded uniformly in $\ell \in \mathcal{L}$ by $M\sqrt{nh^5}\frac{c_+^2}{2}\left|\sum_{i=1}^{n} w_{n,i}^+ \left(\frac{R_i - r_0}{h_+}\right)^2\right|$. Since we have

$$\begin{aligned} \sum_{i=1}^{n} w_{n,i}^+ \left(\frac{R_i - r_0}{h_+}\right)^2 &= \frac{(\hat{\vartheta}_2^+)^2 - \hat{\vartheta}_1^+ \hat{\vartheta}_3^+}{\hat{\vartheta}_2^+ \hat{\vartheta}_0^+ - (\hat{\vartheta}_1^+)^2} \\ &= \frac{\vartheta_2^2 - \vartheta_1 \vartheta_3}{\vartheta_2 \vartheta_0 - \vartheta_1^2} + o_{\mathcal{P}}(1), \end{aligned}$$

where $w_{n,i}^+$ and $\hat{\vartheta}_j^+$ are as defined in Appendix A, and the second line follows by Lemma 2 in Fan and Gijbels (1992); for nonnegative finite j,

$$\hat{\vartheta}_j^+ = f_R^+(r_0)\vartheta_j + o_{\mathcal{P}}(1) \tag{E.8}$$

holds where the $\mathcal{P}$-uniform convergence here follows by Assumption D.1 (i.e., $\mathcal{P}$ shares the common marginal distribution of R). Hence, combined with the undersmoothing bandwidth (Assumption D.2 (iii)), the second term in (E.6) is $o_{\mathcal{P}}(1)$.

The conclusion of the lemma is obtained by verifying $\sup_{\{\ell\in\mathcal{L}\}}\left|\sqrt{nh}\sum_{i=1}^n w_{n,i}^+\mathcal{E}_{1,i}(\ell) - \frac{1}{\sqrt{nh}}\sum_{i=1}^n w_i^+\mathcal{E}_{1,i}(\ell)\right| = o_{\mathcal{P}}(1)$. Consider

$$\begin{aligned}
&\sup_{\{\ell\in\mathcal{L}\}}\left|\sqrt{nh}\sum_{i=1}^n w_{n,i}^+\mathcal{E}_{1,i}(\ell) - \frac{1}{\sqrt{nh}}\sum_{i=1}^n w_i^+\mathcal{E}_{1,i}(\ell)\right| \\
\le & c_+^{-1}\underbrace{\left|\frac{\hat{\vartheta}_2^+}{\hat{\vartheta}_2^+\hat{\vartheta}_0^+ - (\hat{\vartheta}_1^+)^2} - \frac{\vartheta_2}{f_R^+(r_0)(\vartheta_2\vartheta_0 - \vartheta_1^2)}\right|}_{\text{(i)}} \cdot \underbrace{\sup_{\{\ell\in\mathcal{L}\}}\left|\frac{1}{\sqrt{nh}}\sum_{i=1}^n K\left(\frac{R_i - r_0}{h_+}\right)1\{R_i \ge r_0\}\mathcal{E}_{1,i}(\ell)\right|}_{\text{(ii)}} \\
& + c_+^{-1}\underbrace{\left|\frac{\hat{\vartheta}_1^+}{\hat{\vartheta}_2^+\hat{\vartheta}_0^+ - (\hat{\vartheta}_1^+)^2} - \frac{\vartheta_1}{f_R^+(r_0)(\vartheta_2\vartheta_0 - \vartheta_1^2)}\right|}_{\text{(iii)}} \cdot \underbrace{\sup_{\{\ell\in\mathcal{L}\}}\left|\frac{1}{\sqrt{nh}}\sum_{i=1}^n K\left(\frac{R_i - r_0}{h_+}\right)\left(\frac{R_i - r_0}{h_+}\right)1\{R_i \ge r_0\}\mathcal{E}_{1,i}(\ell)\right|}_{\text{(iv)}}.
\end{aligned} \tag{E.9}$$

Since (E.8) implies both terms (i) and (iii) in (E.9) are $o_{\mathcal{P}}(1)$, it suffices to show that the terms (ii) and (iv) in (E.9) are stochastically bounded uniformly in $P \in \mathcal{P}$. Let j be a nonnegative integer and

$$f_{n,i}^{(j)}(\ell) \equiv \frac{1}{\sqrt{h}}K\left(\frac{R_i - r_0}{h_+}\right)\left(\frac{R_i - r_0}{h_+}\right)^j 1\{R_i \ge r_0\}\mathcal{E}_{1,i}(\ell).$$

Consider obtaining a $\mathcal{P}$-uniform bound for $P(\sqrt{n}\sup_{\{\ell\in\mathcal{L}\}}\left|\frac{1}{n}\sum_{i=1}^{n}f_{n,i}^{(j)}(\ell)\right|>\epsilon)$ for $\epsilon>0$ (i.e., term (ii) corresponds to $j=0$ and term (iv) corresponds to $j=1$). By Markov's inequality,

$$\begin{aligned}
P\left(\sqrt{n}\sup_{\{\ell\in\mathcal{L}\}}\left|\frac{1}{n}\sum_{i=1}^{n}f_{n,i}^{(j)}(\ell)\right|>\epsilon\right) &\leq \epsilon^{-1}\sqrt{n}\mathbb{E}_P\left[\sup_{\{\ell\in\mathcal{L}\}}\left|\frac{1}{n}\sum_{i=1}^{n}f_{n,i}^{(j)}(\ell)\right|\right] \\
&= \epsilon^{-1}\sqrt{n}\left(\mathbb{E}_P\left[\max\left\{\sup_{\{\ell\in\mathcal{L}\}}\frac{1}{n}\sum_{i=1}^{n}f_{n,i}^{(j)}(\ell),\ \sup_{\{\ell\in\mathcal{L}\}}\frac{1}{n}\sum_{i=1}^{n}(-f_{n,i}^{(j)}(\ell))\right\}\right]\right) \\
&= \epsilon^{-1}\sqrt{n}\mathbb{E}_P\left[\sup_{\{f_{n,i}^{(j)}\in\mathcal{F}_n^+\cup\mathcal{F}_n^-\}}\frac{1}{n}\sum_{i=1}^{n}f_{n,i}^{(j)}\right], \qquad \text{(E.10)}
\end{aligned}$$

where $\mathcal{F}_n^+\equiv\{f_{n,i}^{(j)}(\ell):\ell\in\mathcal{L}\}$ and $\mathcal{F}_n^-\equiv\{-f_{n,i}^{(j)}(\ell):\ell\in\mathcal{L}\}$. Note that $\mathcal{F}_n^+$ and $\mathcal{F}_n^-$ are VC-subgraph classes whose VC-dimensions are equal to 2 (see, e.g., Lemma A.1 in Kitagawa and Tetenov (2018)) with a uniform envelope $\bar{K}/\sqrt{h}$ and an $L_2(P)$ envelope,

$$\sup_{\{\ell\in\mathcal{L}\}}\|f_{n,i}^{(j)}(\ell)\|_{L_2(P)}\leq\left[c_+\bar{f}_R\int_0^\infty K^2(u)u^{2j}du\right]^{1/2}<\infty.$$

Since $\mathcal{F}_n^+\cup\mathcal{F}_n^-$ is a VC-subgraph class sharing the same uniform and $L_2(P)$ envelope, a maximal inequality for the VC-subgraph class of functions with bounded $L_2(P)$-envelope (Lemma A.5 in Kitagawa and Tetenov (2018)) applies and (E.10) can be bounded from above by

$$C_1\left(c_+\bar{f}_R\int_0^\infty K^2(u)u^{2j}du\right)^{1/2}n^{-1/2}$$

for all n satisfying $nh\geq\frac{C_2\bar{K}^2}{c_+\bar{f}_R\int_0^\infty K^2(u)u^{2j}du}$, where C_1 and C_2 are positive constants that do not depend on P or bandwidth. Since $nh\to\infty$, this maximal inequality with $j=0$ and $j=1$ imply term (ii) and term (iv) in (E.9) are stochastically bounded $\mathcal{P}$-uniformly. Hence,

$$\sup_{\{\ell\in\mathcal{L}\}}\left|\sqrt{nh}\sum_{i=1}^{n}w_{n,i}^+\mathcal{E}_{1,i}(\ell)-\frac{1}{\sqrt{nh}}\sum_{i=1}^{n}w_i^+\mathcal{E}_{1,i}(\ell)\right|=o_{\mathcal{P}}(1) \qquad \text{(E.11)}$$

holds. □

The next lemma shows $\mathcal{P}$-uniform convergence of the covariance kernel of $w_i^\star\mathcal{E}_{d,i}(\cdot)$, the summand in the Bahadur representation of Lemma E.2.

Lemma E.3. *Let $d = 1$ or 0, and $\star = +$ or $-$. For $\ell_1, \ell_2 \in \mathcal{L}$, define*

$$\hat{h}_{2,P,d,\star}(\ell_1, \ell_2) = \frac{1}{nh}\sum_{i=1}^{n}(w_i^{\star})^2\sigma_{P,d}(\ell_1, \ell_2|R_i).$$

Let $\mathcal{P}$ be a class of data generating processes satisfying Assumption D.1 and assume that the kernel function and the bandwidth satisfy Assumption D.2. Then,

$$\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\hat{h}_{2,P,d,\star}(\ell_1, \ell_2) - h_{2,P,d,\star}(\ell_1, \ell_2)\right| = o_{\mathcal{P}}(1),$$

where $h_{2,P,d,\star}$ is as defined in equation (D.2) above.

Proof. We show the claim for the case of $d = 1$ and $\star = +$. The other cases can be proven similarly. Since

$$\begin{aligned}
&\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\hat{h}_{2,P,1,+}(\ell_1, \ell_2) - h_{2,P,1,+}(\ell_1, \ell_2)\right| \\
\leq &\underbrace{\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\hat{h}_{2,P,1,+}(\ell_1, \ell_2) - \mathbb{E}_P[\hat{h}_{2,P,1,+}(\ell_1, \ell_2)]\right|}_{\text{(v)}} + \underbrace{\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\mathbb{E}_P[\hat{h}_{2,P,1,+}(\ell_1, \ell_2)] - h_{2,P,1,+}(\ell_1, \ell_2)\right|}_{\text{(vi)}},
\end{aligned}$$

we show $\mathcal{P}$-uniform convergences of term (v) and term (vi) separately.

First, by exploiting Assumption D.1, we can obtain a uniform upper bound of term (vi) as follows:

$$\left|\mathbb{E}_P[\hat{h}_{2,P,1,+}(\ell_1, \ell_2)] - h_{2,P,1,+}(\ell_1, \ell_2)\right| \leq \frac{5M\bar{f}_R\int_0^{\infty}(\vartheta_2 - \vartheta_1 u)^2 uK^2(u)du}{(f_R^+(r_0))^2(\vartheta_0\vartheta_2 - \vartheta_1^2)^2}h, \tag{E.12}$$

which converges to zero as $n \to \infty$ since $h \to 0$. Since the marginal distribution of R is common for $\mathcal{P}$, this convergence is uniform in $P \in \mathcal{P}$, so term (vi) is $o_{\mathcal{P}}(1)$.

Regarding term (v), Jensen's inequality bounds its mean by

$$\begin{aligned}
&\mathbb{E}_P\left[\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\hat{h}_{2,P,1,+}(\ell_1, \ell_2) - \mathbb{E}_P[\hat{h}_{2,P,1,+}(\ell_1, \ell_2)]\right|\right] \\
\leq &\frac{1}{[c_+f_R^+(r_0)(\vartheta_0\vartheta_2 - \vartheta_1^2)]^2}\mathbb{E}_P\left[\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\frac{1}{n}\sum_{i=1}^{n}f_{n,i}(\ell_1, \ell_2) - \mathbb{E}_P(f_{n,i}(\ell_1, \ell_2))\right|\right],
\end{aligned} \tag{E.13}$$

where $f_{n,i}(\ell_1, \ell_2) \equiv \frac{1}{h}\left[\vartheta_2 - \vartheta_1\left(\frac{R_i - r_0}{h_+}\right)\right]^2 K^2\left(\frac{R_i - r_0}{h_+}\right)\cdot 1\{R_i \geq r_0\}\mathcal{E}_{1,i}(\ell_1)\mathcal{E}_{1,i}(\ell_2)$. Since $\mathcal{E}_{1,i}(\ell_1)\mathcal{E}_{1,i}(\ell_2)$ can be viewed as the sum of three indicator functions for intervals (indexed by ℓ_1 and ℓ_2), $\{f_{n,i}(\ell_1, \ell_2) :$

$\ell_1, \ell_2 \in \mathcal{L}\}$ is a VC-subgraph class of functions with a uniform envelope $h^{-1}(\vartheta_2 + \vartheta_1)^2 \bar{K}^2$ and $L_2(P)$-envelope,

$$[\mathbb{E}_P(f_{n,i}^2(\ell_1, \ell_2))]^{1/2} \leq \frac{1}{\sqrt{h}} \left[c_+ \bar{f}_R \int_0^\infty (\vartheta_2 - \vartheta_1 u)^4 K^4(u)\, du \right]^{1/2}.$$

Applying Lemma A.5 in Kitagawa and Tetenov (2018), we obtain

$$\mathbb{E}_P \left[\sup_{\{\ell_1, \ell_2 \in \mathcal{L}\}} \left| \frac{1}{n} \sum_{i=1}^n f_{n,i}(\ell_1, \ell_2) - E_P(f_{n,i}(\ell_1, \ell_2)) \right| \right] \leq \frac{C_1}{\sqrt{nh}} \sqrt{c_+ \bar{f}_R \int_0^\infty (\vartheta_2 - \vartheta_1 u)^4 K^4(u)\, du} \tag{E.14}$$

for all $nh \geq \frac{C_2(\vartheta_2+\vartheta_1)^4 \bar{K}^4}{c_+ \bar{f}_R \int_0^\infty (\vartheta_2 - \vartheta_1 u)^4 K^4(u) du}$, where C_1 and C_2 are positive constraints that do not depend on P and h. Combining (E.13), (E.14), and Markov's inequality, we conclude that term (v) is $o_{\mathcal{P}}(1)$. □

Exploiting the preceding two lemmas, the next lemma proves the functional central limit theorem for $\hat{m}_{n,d,\star}$ along sequences of the data generating processes in $\mathcal{P}$.

Lemma E.4. *Suppose that Assumptions D.1 and D.2 hold, and let $\{P_n\}$ be a sequence of data generating processes in $\mathcal{P}$. Then, for any subsequence $\{k_n\}$ of $\{n\}$ such that for $d = 0, 1$ and $\star = +, -$, $\lim_{n \to \infty} d(h_{2, P_{k_n}, d, \star}, h^*_{2,d,\star}) = 0$ for some $h^*_{2,d,\star} \in \mathcal{H}_2$, we have*

$$\sqrt{k_n h}(\hat{m}_{d,\star}(\cdot) - m_{P_{k_n}, d, \star}(\cdot)) \Rightarrow \Phi_{h^*_{2,d,\star}}(\cdot), \tag{E.15}$$

where Φ_{h_2} denotes a mean zero Gaussian process with covariance kernel h_2. In addition, we have for $d = 0, 1$,

$$\sqrt{k_n h}(\hat{\nu}_d(\cdot) - \nu_{P_{k_n}, d}(\cdot)) \Rightarrow \Phi_{h^*_{2,d}}(\cdot),$$

where $h^*_{2,d} = h^*_{2,d,+} + h^*_{2,d,-}$.

Proof. To simplify notation, we show this theorem for a sequence $\{n\}$. All the arguments go through with $\{k_n\}$ in place of $\{n\}$.

By Lemma E.2, (E.15) follows if we show $\frac{1}{\sqrt{nh}} \sum_{i=1}^n w_i^+ \mathcal{E}_{1,i}(\cdot) \Rightarrow \Phi_{h^*_{2,d,+}}(\cdot)$. For this purpose, we apply the functional central limit theorem (FCLT; Theorem 10.6 of Pollard (1990)) to the triangular array of independent processes, $\{f_{n,i}(\cdot) : 1 \leq i \leq n\}$, where $f_{n,i}(\ell) = \frac{1}{\sqrt{nh}} w_i^+ \mathcal{E}_{1,i}(\ell)$, $\ell \in \mathcal{L}$. Let their envelope functions be $\{F_{n,i} : 1 \leq i \leq n\}$ with $F_{n,i} = (nh)^{-1/2} |w_i^+|$. Define empirical processes indexed by $\ell \in \mathcal{L}$ as $\widehat{\Phi}_n^+(\ell) = \sum_{i=1}^n f_{n,i}(\ell)$. First, since $\{f_{n,i}(\ell) : \ell \in \mathcal{L}\}$ is a VC-subgraph class of functions (see, e.g., Lemma A.1 in Kitagawa and Tetenov (2018)), manageability of $\{f_{n,i}(\ell) : \ \ell \in \mathcal{L}, 1 \leq i \leq n\}$ (condition (i) of Theorem 10.6 in Pollard (1990)) is implied by a polynomial bound for the packing number of the VC-subgraph class of functions (see, e.g., Theorem 4.8.1 in Dudley (1999)). For condition (ii) of Theorem 10.6 in Pollard (1990), note that

$$
\begin{aligned}
\mathbb{E}_{P_n}[\widehat{\Phi}_n^+(\ell_1)\widehat{\Phi}_n^+(\ell_2)] &= \frac{1}{h}\mathbb{E}_{P_n}\left[(w_i^+)^2\mathcal{E}_{1,i}(\ell_1)\mathcal{E}_{1,i}(\ell_2)\right] \\
&= \mathbb{E}_{P_n}[\hat{h}_{2,P_n,1,+}(\ell_1,\ell_2)] = h_{2,P_n,1,+}(\ell_1,\ell_2) + o(1) \\
&\to h_{2,1,+}^*(\ell_1,\ell_2),
\end{aligned}
$$

as $n \to \infty$, where the $o(1)$ term in the second line follows from the bound shown in (E.12), and the third line follows by the assumption on $\{P_n\}$ in the current lemma. Condition (iii) of Theorem 10.6 in Pollard (1990) can be shown by noting

$$
\sum_{i=1}^n \mathbb{E}_{P_n}[F_{n,i}^2] = \frac{1}{h}\mathbb{E}_{P_n}[(w_i^+)^2] \leq \frac{\bar{f}_R \int_0^\infty (\vartheta_2 - \vartheta_1 u)^2 K^2(u)du}{c_+(f_R^+(r_0))^2(\vartheta_2\vartheta_0 - \vartheta_1^2)^2}.
$$

Condition (iv) of Theorem 10.6 in Pollard (1990) follows by that, for any $\epsilon > 0$,

$$
\begin{aligned}
\sum_{i=1}^n \mathbb{E}_{P_n}[F_{n,i}^2 \cdot 1\{F_{n,i} > \epsilon\}] &\leq \sum_{i=1}^n \mathbb{E}_{P_n}\left[\frac{F_{n,i}^4}{\epsilon^2}\right] = \frac{1}{\epsilon^2 nh^2}E_{P_n}[(w_i^+)^4] \\
&\leq (nh)^{-1}\frac{\bar{f}_R \int_0^\infty (\vartheta_2 - \vartheta_1 u)^4 K^4(u)du}{\epsilon^2 c_+^3[f_R^+(r_0)(\vartheta_0\vartheta_2 - \vartheta_1^2)]^4} \to 0 \ \text{ as } n \to \infty,
\end{aligned}
$$

where the first inequality holds because $1\{F_{n,i} > \epsilon\} \leq (F_{n,i}/\epsilon)^\varsigma$ for any $\varsigma > 0$ and we take $\varsigma = 2$ here.

To show condition (v) of Theorem 10.6 in Pollard (1990), note that

$$
\begin{aligned}
\hat{\rho}_{1,+}^2(\ell_1,\ell_2) &= \sum_{i=1}^n E_{P_n}(f_{n,i}(\ell_1) - f_{n,i}(\ell_2))^2 \\
&= h_{2,P_n,1,+}(\ell_1,\ell_1) - 2h_{2,P_n,1,+}(\ell_1,\ell_2) + h_{2,P_n,1,+}(\ell_2,\ell_2) + o(1) \\
&\to h_{2,1,+}^*(\ell_1,\ell_1) - 2h_{2,1,+}^*(\ell_1,\ell_2) + h_{2,1,+}^*(\ell_2,\ell_2) \equiv \rho_{1,+}^2(\ell_1,\ell_2),
\end{aligned}
$$

where the second line follows by (E.12). Note that the convergence in the last line holds uniformly over $\ell_1, \ell_2 \in \mathcal{L}$ by Lemma E.3, and this uniform convergence ensures condition (v) of Theorem 10.6 in Pollard (1990).

Hence, by the FCLT of Pollard (1990), we obtain $\sqrt{nh}(\hat{m}_{1,+}(\ell) - m_{P_n,1,+}(\ell)) \Rightarrow \Phi_{h_{2,1,+}}(\ell)$. Similarly, we can show $\sqrt{nh}(\hat{m}_{1,-}(\ell) - m_{P_n,1,-}(\ell)) \Rightarrow \Phi_{h_{2,1,-}}(\ell)$.

To show the second part, note that

$$\begin{aligned}&\sqrt{nh}(\hat{v}_1(\ell) - v_{P_n,1}(\ell)) = \sqrt{nh}(\hat{m}_{1,-}(\ell) - m_{P_n,1,-}(\ell)) - \sqrt{nh}(\hat{m}_{1,+}(\ell) - m_{P_n,1,+}(\ell))\\ \Rightarrow &\Phi_{h^*_{2,1,-}+h^*_{2,1,+}}(\ell) = \Phi_{h^*_{2,1}}(\ell),\end{aligned}$$

where the weak convergence holds due to the fact that $\hat{m}_{n,1,+}(\ell)$ and $\hat{m}_{n,1,-}(\ell)$ are estimated from separate samples, so that the two processes are mutually independent. The same arguments apply to the $d = 0$ case. This completes the proof. □

Define, for $d = 1, 0$ and $\star = +, -$,

$$\widehat{\Phi}^u_{n,d,\star}(\ell) = \sum_{i=1}^n U_i \cdot \sqrt{nh} w^\star_{n,i}(g_\ell(Y_i) D_i^d (1 - D_i)^{1-d} - \hat{m}_{n,d,\star}(\ell)).$$

We denote weak convergence conditional on a sample generated from a sample size-dependent distribution of data P_n by $\overset{P_n}{\Rightarrow}$.[4] We denote convergence in probability along the sequence $\{P_n\}$ by $\overset{P_n}{\to}$.

Lemma E.5. *Suppose that Assumptions D.1-D.3 hold, and let $\{P_n\}$ be a sequence of data generating processes in $\mathcal{P}$. For a subsequence $\{k_n\}$ of $\{n\}$ such that for $d = 0, 1$ and $\star = +, -$, $\lim_{n\to\infty} d(h_{2,P_{k_n},d,\star}, h^*_{2,d,\star}) = 0$ holds for some $h^*_{2,d,\star} \in \mathcal{H}_2$, then $\widehat{\Phi}^u_{k_n,d,\star} \overset{P_{k_n}}{\Rightarrow} \Phi_{h^*_{2,d,\star}}$. In addition, for $d = 0, 1$, $\widehat{\Phi}^u_{v_1,k_n} \equiv \widehat{\Phi}^u_{n,1,-}(\ell) - \widehat{\Phi}^u_{n,1,+}(\ell) \overset{P_{k_n}}{\Rightarrow} \Phi_{h^*_{2,d}}$ and $\widehat{\Phi}^u_{v_0,k_n} \equiv \widehat{\Phi}^u_{n,0,+}(\ell) - \widehat{\Phi}^u_{n,0,-}(\ell) \overset{P_{k_n}}{\Rightarrow} \Phi_{h^*_{2,d}}$ hold with $h^*_{2,d} = h^*_{2,d,+} + h^*_{2,d,-}$ defined in (D.2).*

Proof. To simplify notation, we show this theorem for a sequence $\{n\}$, since all the arguments go through with $\{k_n\}$ in place of $\{n\}$. For the first part, it is sufficient to show the case of $\widehat{\Phi}^u_{n,1,+}$ since

[4] Extending the definition of conditional weak convergence given in Section 2.9 of Van Der Vaart and Wellner (1996) to a sequence of data distributions, $\hat{\Phi}^u_n \overset{P_n}{\Rightarrow} \Phi$ means for any $\epsilon > 0$, $\lim_{n\to\infty} P_n(\sup_{\{f\in BL\}} |E_u(f(\hat{\Phi}^u_n)) - E(f(\Phi))| > \epsilon) = 0$, where f maps random element $\Phi(\cdot)$ to $\mathbb{R}$, BL collects f with a bounded Lipschitz constant, and $E_u(\cdot)$ is the expectation of $(U_i : i = 1, \ldots, n)$ conditional on the data.

the arguments for the other cases are the same. We use the same arguments as the proof in Hsu (2016). We define $\hat{\phi}_{n,i,1,+}(\ell) = \sqrt{nh} w^{+}_{n,i}(g_\ell(Y_i)D_i - \hat{m}_{1,+}(\ell))$, so $\widehat{\Phi}^u_{n,1,+} = \sum_{i=1}^n U_i \cdot \hat{\phi}_{n,i,1,+}(\ell)$.

First, we note that the triangular array $\{\hat{f}_{n,i}(\ell) = U_i \cdot \hat{\phi}_{n,i,1,+}(\ell) : \ell \in \mathcal{L}, 1 \leq i \leq n\}$ is manageable with respect to envelope functions $\{\widehat{F}_{n,i} = 2\sqrt{nh}|U_i| \cdot |w^{+}_{n,i}| : 1 \leq i \leq n\}$. Define $\hat{h}_{2,1,+}(\ell_1, \ell_2) = \sum_{i=1}^n \hat{\phi}_{n,i,1,+}(\ell_1)\hat{\phi}_{n,i,1,+}(\ell_2)$. If we have

$$\sup_{\{\ell_1,\ell_2 \in \mathcal{L}\}} |\hat{h}_{2,1,+}(\ell_1, \ell_2) - h^*_{2,1,+}(\ell_1, \ell_2)| \xrightarrow{P_n} 0, \tag{E.16}$$

and

$$nh \sum_{i=1}^n |w^{+}_{n,i}|^2 \xrightarrow{P_n} M_1, \tag{E.17}$$

$$n^3 h^3 \sum_{i=1}^n |w^{+}_{n,i}|^4 \xrightarrow{P_n} M_2, \tag{E.18}$$

for $M_1, M_2 < \infty$, adopting the proof of Theorem 2.1 of Hsu (2016) yields $\widehat{\Phi}^u_{n,1,+}(\ell) \overset{P_u}{\Rightarrow} \Phi_{h^*_{2,1,+}}(\ell)$, and similarly for $\widehat{\Phi}^u_{n,1,-}(\ell) \overset{P_u}{\Rightarrow} \Phi_{h^*_{2,1,-}}(\ell)$. For the second part, note that $\widehat{\Phi}^u_{\nu_1,n}(\ell) = \widehat{\Phi}^u_{n,1,-}(\ell) - \widehat{\Phi}^u_{n,1,+}(\ell)$ and by the independence of the two simulated processes, we have $\widehat{\Phi}^u_{\nu_1,n}(\ell) \overset{P_u}{\Rightarrow} \Phi_{h^*_{2,1}}(\ell)$.

Hence, the rest of the proof focuses on verifying (E.16) - (E.18). For positive integer $j < \infty$ and nonnegative integer $k < \infty$, a straightforward extension of Lemma 2 in Fan and Gijbels (1992) gives

$$(nh)^{(j-1)} \sum_{i=1}^n |w^{+}_{n,i}|^j \left(\frac{R_i - r_0}{h_+}\right)^k = \frac{\int_0^\infty K^j(u)(\vartheta_2 - \vartheta_1 u)^j u^k du}{c_+^{j-1}[\vartheta_0\vartheta_2 - \vartheta_1^2]^j} + o_{\mathcal{P}}(1), \tag{E.19}$$

where the first term on the right-hand side is finite, and the assumption that $\mathcal{P}$ shares a fixed distribution for R leads to this convergence being uniform over $\mathcal{P}$. Hence, (E.17) and (E.18) hold, as $\{P_n\} \in \mathcal{P}$.

To show (E.16), it suffices to show

$$\begin{aligned}
&\sup_{\{\ell_1,\ell_2 \in \mathcal{L}\}} |\hat{h}_{2,1,+}(\ell_1, \ell_2) - h_{2,P,1,+}(\ell_1, \ell_2)| \\
\leq &\sup_{\{\ell_1,\ell_2 \in \mathcal{L}\}} |\hat{h}_{2,1,+}(\ell_1, \ell_2) - \mathbb{E}_P[\hat{h}_{2,P,1,+}(\ell_1, \ell_2)]| + \sup_{\{\ell_1,\ell_2 \in \mathcal{L}\}} |\mathbb{E}_P[\hat{h}_{2,P,1,+}(\ell_1, \ell_2)] - h_{2,P,1,+}(\ell_1, \ell_2)| \qquad \text{(E.20)} \\
= &o_{\mathcal{P}}(1).
\end{aligned}$$

The proof of Lemma E.3 shows that the second term in (E.20) converges to zero uniformly in $\mathcal{P}$. We hence focus on showing that the first term in (E.20) is $o_{\mathcal{P}}(1)$.

Rewrite $\hat{\phi}_{n,i,1,+}(\ell)$ as follows by applying the mean value expansion:

$$\begin{aligned}\hat{\phi}_{n,i,1,+}(\ell) &= \sqrt{nh}w_{n,i}^{+}\left[m_{P,1}(\ell,R_i) - \hat{m}_{1,+}(\ell) + \mathcal{E}_{1,i}(\ell)\right] \\ &= w_{n,i}^{+}\hat{a}_1(\ell) + \hat{a}_{2,i}(\ell) + \hat{a}_{3,i}(\ell),\end{aligned}$$

where

$$\begin{aligned}\hat{a}_1(\ell) &\equiv -\sqrt{nh}[\hat{m}_{1,+}(\ell) - m_{P,1,+}(\ell)] \\ \hat{a}_{2,i}(\ell) &\equiv \sqrt{nh}w_{n,i}^{+}\left[h_+ m_{P,+}^{(1)}(\ell,R_i)\left(\frac{R_i - r_0}{h_+}\right) + \frac{h_+^2}{2}m_{P,1}^{(2)}(\ell,\tilde{R}_i)\left(\frac{R_i - r_0}{h_+}\right)^2\right], \\ \hat{a}_{3,i}(\ell) &\equiv \sqrt{nh}w_{n,i}^{+}\mathcal{E}_{1,i}(\ell).\end{aligned}$$

Then, we have

$$\begin{aligned}\hat{h}_{2,1,+}(\ell_1,\ell_2) =& \underbrace{\hat{a}_1(\ell_1)\hat{a}_1(\ell_2)\sum_{i=1}^{n}(w_{n,i}^{+})^2}_{\text{(i)}} + \underbrace{\sum_{i=1}^{n}\hat{a}_{2,i}(\ell_1)\hat{a}_{2,i}(\ell_2)}_{\text{(ii)}} + \underbrace{\sum_{i=1}^{n}\hat{a}_{3,i}(\ell_1)\hat{a}_{3,i}(\ell_2)}_{\text{(iii)}} \\ &+ \underbrace{\sum_{i=1}^{n}w_{n,i}^{+}\left[\hat{a}_1(\ell_1)(\hat{a}_{2,i}(\ell_2) + \hat{a}_{3,i}(\ell_2)) + \hat{a}_1(\ell_2)(\hat{a}_{2,i}(\ell_1) + \hat{a}_{3,i}(\ell_1))\right]}_{\text{(iv)}} \\ &+ \underbrace{\sum_{i=1}^{n}\left[\hat{a}_{2,i}(\ell_1)\hat{a}_{3,i}(\ell_2) + \hat{a}_{2,i}(\ell_2)\hat{a}_{3,i}(\ell_2)\right]}_{\text{(v)}}.\end{aligned}$$

By Lemma E.4 and (E.19), term (i) is $o_{\mathcal{P}}(1)$ uniformly over $\ell_1, \ell_2 \in \mathcal{L}$. By Assumption D.1 (v), the absolute value of term (ii) can be bounded by $\left\{2M(nh)\sum_{i=1}^{n}(w_{n,i}^{+})^2\left[\left(\frac{R_i - r_0}{h_+}\right) + \left(\frac{R_i - r_0}{h_+}\right)^2\right]\right\} \cdot (h_+ \vee h_+^2)$ uniformly over $\ell_1, \ell_2 \in \mathcal{L}$, which is $o_{\mathcal{P}}(1)$ by (E.19) and $h_+ \to 0$. To examine term (iv), note that

$$
\begin{aligned}
&\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\sum_{i=1}^{n} w_{n,i}^{+}\hat{a}_1(\ell_1)\hat{a}_{2,i}(\ell_2)\right| \\
\leq&(nh)^{-1/2}\sup_{\{\ell\in\mathcal{L}\}}|\hat{a}_1(\ell)|\cdot 2M(nh)\sum_{i=1}(w_{n,i}^{+})^2\left|\left(\frac{R_i-r_0}{h_+}\right)+\left(\frac{R_i-r_0}{h_+}\right)^2\right|\cdot(h_+\vee h_+^2) \\
=&o_{\mathcal{P}}(1),
\end{aligned}
$$

where the final line follows by Lemma E.4, equation (E.19), $nh\to\infty$, and $h_+\to 0$. Note also that

$$
\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\sum_{i=1}^{n} w_{n,i}^{+}\hat{a}_1(\ell_1)\hat{a}_{3,i}(\ell_2)\right|\leq(nh)^{-1}\sup_{\{\ell\in\mathcal{L}\}}|\hat{a}_1(\ell)|\cdot\sup_{\{\ell\in\mathcal{L}\}}\left|(nh)^{3/2}\sum_{i=1}^{n}(w_{n,i}^{+})^2\mathcal{E}_{1,i}(\ell)\right|.
$$

The proof of (E.11) in Lemma E.2 can be extended to claim the following Bahadur representation:

$$
\sup_{\{\ell\in\mathcal{L}\}}\left|(nh)^{3/2}\sum_{i=1}^{n}(w_{n,i}^{+})^2\mathcal{E}_{1,i}(\ell)-\frac{1}{\sqrt{nh}}\sum_{i=1}^{n}(w_i^{+})^2\mathcal{E}_{1,i}(\ell)\right|=o_{\mathcal{P}}(1).
$$

As in the proof of Lemma E.4, FCLT applied to $\frac{1}{\sqrt{nh}}\sum_{i=1}^{n}(w_i^{+})^2\mathcal{E}_{1,i}(\ell)$ shows $\sup_{\{\ell\in\mathcal{L}\}}\left|(nh)^{3/2}\sum_{i=1}^{n}(w_{n,i}^{+})^2\mathcal{E}_{1,i}(\ell)\right|$ is stochastically bounded uniformly in $\mathcal{P}$. Combining this with Lemma E.4 and $nh\to\infty$, we obtain $\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\sum_{i=1}^{n} w_{n,i}^{+}\hat{a}_1(\ell_1)\hat{a}_{3,i}(\ell_2)\right|=o_{\mathcal{P}}(1)$. This implies term (iv) is $o_{\mathcal{P}}(1)$. Regarding term (v), we have

$$
\begin{aligned}
&\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\sum_{i=1}^{n}\hat{a}_{2,i}(\ell_1)\hat{a}_{3,i}(\ell_2)\right| \\
\leq&M(nh)^{-1/2}(h_+\vee h_+^2)\times\left\{\sup_{\{\ell\in\mathcal{L}\}}\left|(nh)^{3/2}\sum_{i=1}(w_{n,i}^{+})^2\left(\frac{R_i-r_0}{h_+}\right)\mathcal{E}_{1,i}(\ell)\right|\right. \\
&\left.+\sup_{\{\ell\in\mathcal{L}\}}\left|(nh)^{3/2}\sum_{i=1}(w_{n,i}^{+})^2\left(\frac{R_i-r_0}{h_+}\right)^2\mathcal{E}_{1,i}(\ell)\right|\right\}.
\end{aligned}
\tag{E.21}
$$

Similar to the proof of (E.11) in Lemma E.2, the two terms in the curly brackets of (E.21) admit the following Bahadur representation: for positive integer $j<\infty$,

$$
\sup_{\{\ell\in\mathcal{L}\}}\left|(nh)^{3/2}\sum_{i=1}^{n}(w_{n,i}^{+})^2\left(\frac{R_i-r_0}{h_+}\right)^j\mathcal{E}_{1,i}(\ell)-\frac{1}{\sqrt{nh}}\sum_{i=1}^{n}(w_i^{+})^2\left(\frac{R_i-r_0}{h_+}\right)^j\mathcal{E}_{1,i}(\ell)\right|=o_{\mathcal{P}}(1).
$$

Similar to the proof of Lemma E.4, the FCLT applied to $\frac{1}{\sqrt{nh}}\sum_{i=1}^{n}(w_i^+)^2\left(\frac{R_i-r_0}{h_+}\right)^j\mathcal{E}_{1,i}(\ell)$ shows that it is stochastically bounded uniformly in $\mathcal{P}$. Accordingly, since $nh\to\infty$ and $h_+\to 0$, the upper bound in (E.21) is $o_{\mathcal{P}}(1)$.

We now show term (iii) is the leading term such that $\sup_{\{\ell_1,\ell_2\}}\left|\sum_{i=1}^{n}\hat{a}_{3,i}(\ell_1)\hat{a}_{3,i}(\ell_2)-\mathbb{E}_P[\hat{h}_{2,P,1,+}(\ell_1,\ell_2)]\right| = o_{\mathcal{P}}(1)$ holds. Modifying the proof of (E.11) by replacing $f_{n,i}^j(\ell)$ with $f_{n,i}(\ell_1,\ell_2)$ defined in the proof of Lemma E.3, we obtain the Bahadur-type uniform approximation,

$$\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\sum_{i=1}^{n}\hat{a}_{3,i}(\ell_1)\hat{a}_{3,i}(\ell_2)-(nh)^{-1}\sum_{i=1}^{n}(w_i^+)^2\mathcal{E}_{1,i}(\ell_1)\mathcal{E}_{1,i}(\ell_2)\right| = o_{\mathcal{P}}(1).$$

We hence aim to verify $\left|(nh)^{-1}\sum_{i=1}^{n}(w_i^+)^2\mathcal{E}_{1,i}(\ell_1)\mathcal{E}_{1,i}(\ell_2)-\mathbb{E}_P[\hat{h}_{2,P,1,+}(\ell_1,\ell_2)]\right| = o_{\mathcal{P}}(1)$. Note that

$$\begin{aligned}&\mathbb{E}_P\left[\left|(nh)^{-1}\sum_{i=1}^{n}(w_i^+)^2\mathcal{E}_{1,i}(\ell_1)\mathcal{E}_{1,i}(\ell_2)-\mathbb{E}_P[\hat{h}_{2,P,1,+}(\ell_1,\ell_2)]\right|\right]\\ \le&\frac{1}{[c_+f_R^+(r_0)(\vartheta_0\vartheta_2-\vartheta_1^2)]^2}\mathbb{E}_P\left[\sup_{\{\ell_1,\ell_2\in\mathcal{L}\}}\left|\frac{1}{n}\sum_{i=1}^{n}f_{n,i}(\ell_1,\ell_2)-\mathbb{E}_P(f_{n,i}(\ell_1,\ell_2))\right|\right],\end{aligned} \tag{E.22}$$

where $f_{n,i}(\ell_1,\ell_2)$ is as defined in the proof of Lemma E.3. Note that this upper bound coincides with (E.13). Hence, the proof of Lemma E.3 yields $\left|(nh)^{-1}\sum_{i=1}^{n}(w_i^+)^2\mathcal{E}_{1,i}(\ell_1)\mathcal{E}_{1,i}(\ell_2)-\mathbb{E}_P[\hat{h}_{2,P,1,+}(\ell_1,\ell_2)]\right| = o_{\mathcal{P}}(1)$. □

Lemma E.6. *Suppose that Assumptions D.1 and D.2 hold. Let $\{P_n\}$ be a sequence of data generating processes in $\mathcal{P}$. For any subsequence $\{k_n\}$ of $\{n\}$ such that for $d=0,1$ and $\star=+,-$, $\lim_{n\to\infty}d(h_{2,P_{k_n},d,\star},h^*_{2,d,\star})=0$ for some $h^*_{2,d,\star}\in\mathcal{H}_2$, then for $d=0,1$, $\sup_{\{\ell\in\mathcal{L}\}}|\hat{\sigma}^{-1}_{d,\xi}(\ell)-\sigma^{-1}_{d,P_{k_n},\xi}(\ell)|\xrightarrow{P_{k_n}}0$, where $\sigma_{d,P_{k_n},\xi}(\ell)\equiv\max\{h_{2,P_{k_n},d}(\ell,\ell),\xi\}$.*

Proof. Using the notation defined in the proof of Lemma E.5, we note, for $d=0,1$,

$$\hat{\sigma}_{d,\xi}(\ell)=\max\{\xi,\sqrt{\hat{h}^2_{2,d,+}(\ell,\ell)+\hat{h}^2_{2,d,-}(\ell,\ell)}\}.$$

The uniform convergence of (E.20) shown in the proof of Lemma E.5 implies

$$\sup_{\{\ell \in \mathcal{L}\}} \left| \sqrt{\hat{h}^2_{2,d,+}(\ell,\ell) + \hat{h}^2_{2,d,-}(\ell,\ell)} - h_{2,P_{k_n},d}(\ell,\ell) \right| \xrightarrow{P_{k_n}} 0.$$

Due to the fact that the maximum operator is a continuous functional and the fact that $\sigma_{d,k_n,\xi}$ is bounded away from zero, $\sup_{\{\ell \in \mathcal{L}\}} |\hat{\sigma}^{-1}_{d,\xi}(\ell) - \sigma^{-1}_{d,P_{k_n},\xi}(\ell)| \xrightarrow{P_{k_n}} 0$ follows by the continuous mapping theorem. □

Remark: Note that the results in Lemmas E.4 and E.5 hold jointly for $d = 0$ and $d = 1$. We omit the results and proofs for brevity.

Proof of Theorem D.1: Having shown Lemmas E.4 to E.6, we prove the current theorem adapting the proof of Theorem 2 in Andrews and Shi (2013). Let $\mathcal{H}_1$ denote the set of measurable functions mapping $\mathcal{L}$ to $[-\infty, 0]$. Let $h = (h_1, h_2)$, where $h_1 = (h_{1,0}, h_{1,1}) \in \mathcal{H}_1 \times \mathcal{H}_1$ and $h_2 = (h_{2,0}, h_{2,1}) \in \mathcal{H}_2 \times \mathcal{H}_2$. Define

$$T(h) = \sup_{\{d \in \{0,1\}, \ell \in \mathcal{L}\}} \frac{\Phi_{h_{2,d}}(\ell)}{\sigma_{d,P_{k_n},\xi}(\ell)} + h_{1,d}(\ell).$$

Define $c_0(h_1, h_2, \alpha)$ as the (1-α)-th quantile of $T(h)$. Similar to Lemma A2 of Andrews and Shi (2013), we can show that for any $\zeta > 0$,

$$\limsup_{n\to\infty} \sup_{\{P \in \mathcal{P}_0:\ d \in \{0,1\}, h_{2,P,d,+}, h_{2,P,d,-} \in \mathcal{H}_{2,cpt}\}} P\Big(\widehat{S}_n > c_0(h^P_{1,n}, h_{2,P}, \alpha) + \zeta\Big) \leq \alpha, \tag{E.23}$$

where $h^P_{1,n} = (h^P_{1,0,n}, h^P_{1,1,n})$ such that for $d = 0, 1$, $h^P_{1,d,n} = \sqrt{nh}\nu_{P,d}$ which belongs to $\mathcal{H}_1$ under $P \in \mathcal{P}_0$. Also, similar to Lemma A3 of Andrews and Shi (2013), we can show that for all $\alpha < 1/2$

$$\limsup_{n\to\infty} \sup_{\{P \in \mathcal{P}_0:\ d \in \{0,1\}, h_{2,P,d,+}, h_{2,P,d,-} \in \mathcal{H}_{2,cpt}\}} P\Big(c_0(\psi_n, h_{2,P}, \alpha) < c_0(h^P_{1,n}, h_{2,P}, \alpha)\Big) = 0, \tag{E.24}$$

where $\psi_n(\ell) = (\psi_{n,0}(\ell), \psi_{n,1}(\ell))$, $\ell \in \mathcal{L}$, as defined in Algorithm 1 in the main text. To complete the proof, it suffices to show that for all $0 < \zeta < \eta$,

$$\limsup_{n\to\infty} \sup_{\{P \in \mathcal{P}_0:\ d \in \{0,1\}, h_{2,P,d,+}, h_{2,P,d,-} \in \mathcal{H}_{2,cpt}\}} P\Big(\hat{c}_\eta(\alpha) < c_0(\psi_n, h_{2,P}, 1-\alpha) + \zeta\Big) = 0. \tag{E.25}$$

Let $\{P_n \in \mathcal{P}_0 : n \geq 1\}$ be a sequence for which the probability in equation (E.25) evaluated at P_n differs from its supremum over $P \in \mathcal{P}_0$ by $\delta_n > 0$ or less and $\lim_{n\to\infty} \delta_n = 0$. By the definition of $\limsup$, such a sequence always exists. Therefore, it is equivalent to show that for $0 < \zeta < \eta$,

$$\lim_{n\to\infty} P_n\left(\hat{c}_{n,\eta}(\alpha) < c_0(\psi_n, h_{2,P_n}, \alpha) + \zeta\right) = 0, \tag{E.26}$$

where $\hat{c}_{n,\eta}(\alpha)$ is $\hat{c}_\eta(\alpha)$ with its dependence on the sample size along the sequence of sampling distributions $\{P_n\}$ notated explicitly. The limit on the left-hand side of (E.26) exists by the construction of $\{P_n\}$, and we want to show it is equal to 0. Given that we restrict $h_{2,P,d,+}$ and $h_{2,P,d,-}$ to a compact set $\mathcal{H}_{2,cpt}$, there exists a subsequence $\{k_n\}$ of $\{n\}$ such that for $d = 0, 1$, $h_{2,P_{k_n},d,+}$ and $h_{2,P_{k_n},d,-}$ converge to $h^*_{2,d,+}$ and $h^*_{2,d,-}$, respectively, for some $h^*_{2,d,+}, h^*_{2,d,-} \in \mathcal{H}_{2,cpt}$, $d = 0, 1$.

By Lemmas E.4, E.5, and E.6,

$$\begin{aligned}
&\sqrt{k_n h}(\hat{\nu}_d(\cdot) - \nu_{P_{k_n},d}(\cdot)) \Rightarrow \Phi_{h^*_{2,d}}(\cdot), \\
&\widehat{\Phi}^u_{\nu_d,k_n}(\cdot) \overset{P_{k_n}}{\Rightarrow} \Phi'_{h^*_{2,d}}(\cdot), \\
&\sup_{\{\ell\in\mathcal{L}\}} |\hat{\sigma}^{-1}_{d,\xi}(\ell) - \sigma^{-1}_{d,P_{k_n},\xi}(\ell)| \overset{P_{k_n}}{\to} 0
\end{aligned}$$

for $d = 0, 1$, where $\Phi'_{h^*_{2,d}}(\ell)$ is an independent copy of $\Phi_{h^*_{2,d}}(\ell)$. By the almost sure representation theorem (e.g., Theorem 9.4 of Pollard (1990)), there exists a probability space and random objects $(\tilde{\nu}_d(\cdot), \widetilde{\Phi}^u_{\nu_d,k_n}(\cdot), \tilde{\sigma}_{d,\xi}(\cdot))$ and $(\widetilde{\Phi}_{h^*_{2,d}}(\cdot), \widetilde{\Phi}'_{h^*_{2,d}}(\cdot))$, $d = 0, 1$, defined on it, such that they have the same probability distribution as $(\hat{\nu}_d(\cdot), \widehat{\Phi}^u_{\nu_d,k_n}(\cdot), \hat{\sigma}_{d,\xi}(\cdot))$ and $(\Phi_{h^*_{2,d}}(\cdot), \Phi'_{h^*_{2,d}}(\cdot))$, $d = 0, 1$, and satisfy

$$\sup_{d\in\{0,1\},\ell\in\mathcal{L}} \left\| \begin{pmatrix} \sqrt{k_n h}(\tilde{\nu}_d(\cdot) - \nu_{P_{k_n},d}(\cdot)) \\ \widetilde{\Phi}^u_{\nu_d,k_n}(\ell) \\ \tilde{\sigma}_{d,\xi}(\ell) \end{pmatrix} - \begin{pmatrix} \widetilde{\Phi}_{h^*_{2,d}}(\ell) \\ \widetilde{\Phi}'_{h^*_{2,d}}(\ell) \\ \sigma_{d,P_{k_n},\xi}(\ell) \end{pmatrix} \right\| \to 0, \tag{E.27}$$

as $n \to \infty$, a.s. We also define an analogue of $\psi_{n,d}$ as

$$\tilde{\psi}_{k_n,d}(\cdot) = -B_{k_n} \cdot 1\left\{ \frac{\sqrt{k_n h} \cdot \tilde{\nu}_d(\cdot)}{\tilde{\sigma}_{d,\xi}(\ell)} < -a_{k_n} \right\},$$

and let $\tilde{c}_{k_n,\eta}(\alpha)$ be the $(1-\alpha+\eta)$-th quantile of $\sup_{d\in\{0,1\},\ell\in\mathcal{L}} \left\{ \frac{\widetilde{\Phi}^u_{\nu_d,k_n}(\ell)}{\tilde{\sigma}_{d,\xi}(\ell)} + \tilde{\psi}_{k_n,d}(\ell) \right\}$ plus η, which by construction shares the probability law with $\hat{c}_{k_n,\eta}(\alpha)$.

Let Ω_1 be the subset of the sample space such that the convergence of (E.27) holds. Following the proof of Theorem 1 of Andrews and Shi (2013), we can show an inequality analogous to (12.28) in Andrews and Shi (2013); for any sequence $\{\tilde{a}_{k_n}\} \in \mathbb{R}$ that may depend on h_1 and P, and for any $\zeta_1 > 0$,

$$\limsup_{n\to\infty} \sup_{\{h_{1,0},h_{1,1}\in\mathcal{H}_1\}} \mathbb{P}\left(\sup_{\{d\in\{0,1\},\ell\in\mathcal{L}\}} \frac{\widetilde{\Phi}^u_{\nu_d,k_n}(\ell)}{\tilde{\sigma}_{d,\xi}(\ell)} + h_{1,d}(\ell) \le \tilde{a}_{k_n} \right)$$
$$- \mathbb{P}\left(\sup_{\{d\in\{0,1\},\ell\in\mathcal{L}\}} \frac{\widetilde{\Phi}_{h^*_{2,d}}(\ell)}{\sigma_{d,P_{k_n},\xi}(\ell)} + h_{1,d}(\ell) \le \tilde{a}_{k_n} + \zeta_1 \right) \le 0, \qquad \text{(E.28)}$$

where $\mathbb{P}$ denotes the measure of the probability space that $(\tilde{\nu}_{d,k_n}, \widetilde{\Phi}^u_{\nu_d,k_n}(\cdot), \tilde{\sigma}_{d,\xi}(\cdot) : n = 1,2,\ldots,)$ are defined on. By (E.28) and a similar argument to Lemma A5 of Andrews and Shi (2013), we have that for all $0 < \zeta < \zeta_1 < \eta$ and $\omega \in \Omega_1$,

$$\liminf\nolimits_{n\to\infty} \tilde{c}_{k_n,\eta}(\alpha)(\omega) \ge c_0(\tilde{\psi}_{k_n}, h_{2,P_{k_n}}, \alpha) + \zeta_1. \qquad \text{(E.29)}$$

Given that $P(\Omega_1) = 1$, this implies

$$\lim\nolimits_{n\to\infty} \mathbb{P}(\tilde{c}_{k_n,\eta}(\alpha) < c_0(\tilde{\psi}_{k_n}, h_{2,P_{k_n}}, \alpha) + \zeta) = 0. \qquad \text{(E.30)}$$

Since $(\tilde{c}_{k_n,\eta}(\alpha), \tilde{\psi}_{k_n})$ share the probability law with $(\hat{c}_{k_n,\eta}(\alpha), \psi_{k_n})$, we also have

$$\lim\nolimits_{n\to\infty} P_{k_n}(\hat{c}_{k_n,\eta}(\alpha) < c_0(\psi_{k_n}, h_{2,P_{k_n}}, \alpha) + \zeta) = 0. \qquad \text{(E.31)}$$

For any convergent sequence $\{b_n\}$, if there exists a subsequence $\{b_{k_n}\}$ converging to b, then $\{b_n\}$ converges to b as well. Therefore, (E.31) is sufficient for (E.26). Theorem D.1(a) is shown by combining (E.23), (E.24) and (E.25).

We next show Theorem D.1(b). Under Assumption D.5, consider pointwise asymptotics under $P_c \in \mathcal{P}_0$. Similarly to the proof of Proposition 1 of Barrett and Donald (2003) and Lemma 1 of Donald and Hsu (2016), we can show $\widehat{S}_n \xrightarrow{d} \sup_{\{(d,\ell):\, \ell\in\mathcal{L}^o_{P_c,d}\}} \Phi_{h_{2,P_c,d}}(\ell)/\sigma_{d,P_c,\xi}(\ell)$ whose CDF is denoted by $H(a)$. By Tsirel'son (1975), if either $\Phi_{h_{2,P_c,0}}$ restricted to $\mathcal{L}^o_{P_c,0} \times \mathcal{L}^o_{P_c,0}$ or $\Phi_{h_{2,P_c,1}}$ restricted to $\mathcal{L}^o_{P_c,1} \times \mathcal{L}^o_{P_c,1}$ is not a zero function, then $H(a)$ is continuous and strictly increasing for $a \in (0,\infty)$ and $H(0) > 1/2$.

Following the proof of Theorem 2(b) of Andrews and Shi (2013), we can show that $\hat{c}_\eta(\alpha) \xrightarrow{P_c} q_c(1-\alpha+\eta)+\eta$ where $q_c(1-\alpha+\eta)$ denotes the $(1-\alpha+\eta)$-th quantile of $\sup_{(d,\ell):\ \ell\in\mathcal{L}^o_{P_c,d}} \Phi_{h_{2,P_c,d}}(\ell)/\sigma_{d,P_c,\xi}(\ell)$. Because $H(a)$ is continuous at $q_c(1-\alpha)$, we have $\lim_{\eta\to 0} q_c(1-\alpha+\eta)+\eta = q_c(1-\alpha)$. This suffices to show that $\lim_{n\to\infty} P_c(\widehat{S}_n > \hat{c}_\eta(\alpha)) = \alpha$. Combined with the claim of (a) in the current theorem, Theorem D.1(b) holds. □

Proof of Theorem D.2: Under any fixed alternative P_A, the exists (d,ℓ^*) such that $\nu_d(\ell^*) > 0$, so $\widehat{S}_n/\sqrt{nh} \geq \nu_d(\ell^*)/\sigma_{d,P_A,\xi}(\ell^*)$ in probability that implies that $\widehat{S}_n$ will diverge to positive infinity in probability. Also, the $\hat{c}_\eta(\alpha)$ is bounded in probability, so $\lim_{n\to\infty} P(\widehat{S}_n > \hat{c}_\eta(\alpha)) = 1$. □

Proof of Theorem D.3: Define $\mathcal{L}_d^{++} = \{\ell \in \mathcal{L}^o_{P_c,d} : \delta_d(\ell) > 0\}$. For $d = 1,0$, let $\sigma^*_{d,\xi}(\ell) \equiv \max\{\xi, \sqrt{(h^*_{2,d,+}(\ell,\ell))^2 + (h^*_{2,d,-}(\ell,\ell))^2}\}$ be the limiting trimmed variance along the sequence of local alternatives $\{P_n\}$. It can be shown that $\widehat{S}_n \overset{P_\eta}{\Rightarrow} \sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}} (\Phi_{h^*_{2,d}}(\ell)+\delta_d(\ell))/\sigma^*_{d,\xi}(\ell)$ and $\hat{c}_\eta(\alpha) \xrightarrow{P_n} c_\eta+\eta$ where c_η is the $(1-\alpha+\eta)$-th quantile of $\sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}} \Phi_{h^*_{2,d}}(\ell)/\sigma^*_{d,\xi}(\ell)$. Then, the limit of the local power is

$$P(\sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}} (\Phi_{h^*_{2,d}}(\ell)+\delta_d(\ell))/\sigma^*_{d,\xi}(\ell) \geq c_\eta+\eta).$$

We need to consider the following two cases: (a) both $h^*_{2,0}$ restricted to $\mathcal{L}^o_{P_c,0}\times\mathcal{L}^o_{P_c,0}$ and $h^*_{2,1}$ restricted to $\mathcal{L}^o_{P_c,1}\times\mathcal{L}^o_{P_c,1}$ are zero functions and (b) at least one of $h^*_{2,0}$ restricted to $\mathcal{L}^o_{P_c,0}\times\mathcal{L}^o_{P_c,0}$ or $h^*_{2,1}$ restricted to $\mathcal{L}^o_{P_c,1}\times\mathcal{L}^o_{P_c,1}$ is not a zero function.

For case (a), because $h^*_{2,0}$ restricted to $\mathcal{L}^o_{P_c,0}\times\mathcal{L}^o_{P_c,0}$ and $h^*_{2,1}$ restricted to $\mathcal{L}^o_{P_c,1}\times\mathcal{L}^o_{P_c,1}$ are zero functions, then $\sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}} |\Phi_{h^*_{2,d}}(\ell)| \xrightarrow{P_\eta} 0$ and $\widehat{S}_n \xrightarrow{P_\eta} \sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}} \delta_d(\ell)/\sigma^*_{d,\xi}(\ell) > 0$. Also, it is true that $c_\eta+\eta=\eta$ and when $\eta\to 0$, we have $P(\widehat{S}_n > \eta) = 1$ when η is small enough.

For case (b), when at least one of $h^*_{2,0}$ restricted to $\mathcal{L}^o_{P_c,0}\times\mathcal{L}^o_{P_c,0}$ or $h^*_{2,1}$ restricted to $\mathcal{L}^o_{P_c,1}\times\mathcal{L}^o_{P_c,1}$ is not a zero function, then by the continuity of the distribution of $\sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}} (\Phi_{h^*_{2,d}}(\ell) + \delta_d(\ell))/\sigma^*_{d,\xi}(\ell)$ and $\sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}} \Phi_{h^*_{2,d}}(\ell)/\sigma^*_{d,\xi}(\ell)$,

$$\lim_{\eta\to 0} P(\sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}} (\Phi_{h^*_{2,d}}(\ell)+\delta_d(\ell))/\sigma^*_{d,\xi}(\ell) \geq c_\eta+\eta) = P(\sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}} (\Phi_{h^*_{2,d}}(\ell)+\delta_d(\ell))/\sigma^*_{d,\xi}(\ell) \geq c),$$

where c is the $(1-\alpha)$-th quantile of $\sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}}\Phi_{h^*_{2,d}}(\ell)/\sigma^*_{d,\xi}(\ell)$. By assumption, $\delta_d(\ell)$ is nonnegative if $\ell\in\mathcal{L}^o_{P_c,d}$, so $\sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}}(\Phi_{h^*_{2,d}}(\ell)+\delta_d(\ell))/\sigma^*_{d,\xi}(\ell)$ first order stochastically dominates $\sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}}\Phi_{h^*_{2,d}}(\ell)/\sigma^*_{d,\xi}(\ell)$ and it follows that

$$P(\sup_{\{(d,\ell):\ell\in\mathcal{L}^o_{P_c,d}\}}(\Phi_{h^*_{2,d}}(\ell)+\delta_d(\ell))/\sigma^*_{d,\xi}(\ell)\geq c)\geq\alpha.$$

This completes the proof for Theorem D.3. □

Appendix F. Additional Simulation Results for Section 4

In this section, we report additional simulation results. Tables F.1 to F.3 report detailed results for the size properties of our test using data-driven choices of MSE-optimal bandwidths (AI, IK, and CCT). We consider undersmoothing, an MSE-optimal plus RBC implementation, and a CER-optimal plus RBC implementation, respectively. For the CER implementation, we also implement the data-driven plug-in bandwidth (reported in the DPI column of the relevant tables) as suggested by (see Calonico, Cattaneo, and Farrell, 2020, Section 4.2). Tables F.4 to F.6 show the simulated power properties for the four specifications violating the null.

Table F.1. Size Properties (with undersmoothing)

		AI			IK			CCT		
DGP	n	1%	5%	10%	1%	5%	10%	1%	5%	10%
Size1	1000	0.013	0.060	0.112	0.005	0.02	0.061	0.004	0.019	0.054
	2000	0.025	0.071	0.126	0.003	0.025	0.064	0.005	0.034	0.065
	4000	0.018	0.078	0.140	0.003	0.038	0.084	0.006	0.035	0.074
	8000	0.021	0.065	0.120	0.012	0.045	0.095	0.01	0.033	0.085
Size2	1000	0.018	0.063	0.115	0.003	0.014	0.046	0.003	0.012	0.040
	2000	0.014	0.064	0.116	0.008	0.035	0.074	0.006	0.031	0.062
	4000	0.02	0.064	0.119	0.006	0.041	0.074	0.007	0.038	0.080
	8000	0.013	0.06	0.101	0.005	0.039	0.072	0.006	0.036	0.077

TABLE F.2. Size Properties (MSE-optimal + RBC)

DGP	n	AI 1%	AI 5%	AI 10%	IK 1%	IK 5%	IK 10%	CCT 1%	CCT 5%	CCT 10%
Size1	1000	0.006	0.037	0.079	0.004	0.016	0.041	0.003	0.017	0.036
	2000	0.016	0.058	0.108	0.002	0.018	0.043	0.006	0.022	0.051
	4000	0.020	0.067	0.119	0.008	0.049	0.097	0.002	0.022	0.065
	8000	0.020	0.066	0.125	0.007	0.051	0.085	0.008	0.037	0.077
Size2	1000	0.013	0.039	0.083	0.002	0.008	0.034	0.002	0.015	0.029
	2000	0.012	0.051	0.122	0.004	0.033	0.061	0.002	0.021	0.044
	4000	0.013	0.077	0.130	0.010	0.040	0.078	0.007	0.037	0.077
	8000	0.016	0.065	0.125	0.006	0.042	0.090	0.003	0.040	0.085

TABLE F.3. Size Properties (CER-optimal + RBC)

DGP	n	AI+adjustment 1%	AI+adjustment 5%	AI+adjustment 10%	IK+adjustment 1%	IK+adjustment 5%	IK+adjustment 10%	CCT+adjustment 1%	CCT+adjustment 5%	CCT+adjustment 10%	DPI 1%	DPI 5%	DPI 10%
Size1	1000	0.016	0.055	0.11	0.004	0.025	0.056	0.004	0.024	0.049	0.011	0.042	0.071
	2000	0.026	0.076	0.134	0.005	0.022	0.054	0.005	0.024	0.059	0.016	0.044	0.084
	4000	0.018	0.086	0.140	0.004	0.043	0.101	0.006	0.035	0.068	0.016	0.054	0.110
	8000	0.023	0.057	0.113	0.008	0.045	0.09	0.01	0.036	0.073	0.015	0.064	0.121
Size2	1000	0.018	0.067	0.109	0.002	0.015	0.043	0.003	0.014	0.037	0.005	0.034	0.073
	2000	0.013	0.06	0.117	0.009	0.032	0.078	0.005	0.024	0.055	0.016	0.059	0.121
	4000	0.017	0.06	0.118	0.011	0.042	0.083	0.008	0.039	0.078	0.017	0.062	0.115
	8000	0.013	0.057	0.101	0.008	0.044	0.079	0.007	0.035	0.085	0.017	0.057	0.105

TABLE F.4. Power Properties (undersmoothing)

DGP	n	AI 1%	AI 5%	AI 10%	IK 1%	IK 5%	IK 10%	CCT 1%	CCT 5%	CCT 10%
Power1	1000	0.096	0.215	0.303	0.074	0.174	0.244	0.047	0.111	0.183
	2000	0.235	0.439	0.557	0.218	0.403	0.504	0.113	0.256	0.359
	4000	0.561	0.753	0.816	0.541	0.744	0.824	0.373	0.604	0.693
	8000	0.908	0.962	0.977	0.926	0.975	0.987	0.785	0.907	0.94
Power2	1000	0.051	0.122	0.221	0.014	0.061	0.122	0.011	0.052	0.101
	2000	0.1	0.271	0.383	0.065	0.194	0.296	0.03	0.14	0.214
	4000	0.323	0.554	0.678	0.293	0.511	0.647	0.15	0.342	0.438
	8000	0.741	0.885	0.928	0.752	0.888	0.934	0.503	0.732	0.818
Power3	1000	0.052	0.164	0.28	0.037	0.123	0.197	0.023	0.078	0.134
	2000	0.128	0.299	0.432	0.114	0.257	0.393	0.062	0.17	0.261
	4000	0.359	0.573	0.693	0.283	0.51	0.638	0.187	0.383	0.494
	8000	0.758	0.883	0.938	0.724	0.87	0.922	0.526	0.734	0.831
Power4	1000	0.032	0.099	0.175	0.005	0.05	0.089	0.001	0.024	0.053
	2000	0.058	0.172	0.252	0.031	0.123	0.209	0.017	0.06	0.134
	4000	0.119	0.264	0.399	0.106	0.268	0.383	0.043	0.144	0.24
	8000	0.331	0.55	0.673	0.322	0.54	0.656	0.144	0.326	0.458

TABLE F.5. Power Properties (MSE optimal + RBC)

DGP	n	AI 1%	AI 5%	AI 10%	IK 1%	IK 5%	IK 10%	CCT 1%	CCT 5%	CCT 10%
Power1	1000	0.031	0.103	0.18	0.022	0.081	0.153	0.014	0.05	0.089
	2000	0.088	0.221	0.334	0.084	0.207	0.299	0.033	0.115	0.18
	4000	0.277	0.476	0.577	0.267	0.459	0.549	0.154	0.305	0.414
	8000	0.627	0.812	0.87	0.664	0.82	0.877	0.433	0.654	0.754
Power2	1000	0.024	0.086	0.149	0.005	0.023	0.056	0.002	0.022	0.05
	2000	0.04	0.135	0.233	0.029	0.099	0.173	0.009	0.052	0.095
	4000	0.13	0.293	0.406	0.1	0.246	0.363	0.043	0.142	0.226
	8000	0.396	0.624	0.725	0.4	0.622	0.724	0.186	0.391	0.528
Power3	1000	0.035	0.106	0.176	0.012	0.063	0.12	0.006	0.027	0.071
	2000	0.05	0.174	0.289	0.052	0.154	0.247	0.025	0.079	0.147
	4000	0.182	0.361	0.484	0.118	0.289	0.416	0.07	0.183	0.296
	8000	0.492	0.694	0.781	0.419	0.64	0.744	0.261	0.466	0.582
Power4	1000	0.011	0.057	0.114	0.001	0.024	0.053	0.001	0.017	0.038
	2000	0.03	0.118	0.204	0.017	0.06	0.134	0.01	0.042	0.104
	4000	0.066	0.181	0.288	0.043	0.144	0.24	0.016	0.079	0.168
	8000	0.149	0.341	0.478	0.144	0.326	0.458	0.067	0.201	0.311

TABLE F.6. Power Property (CER optimal + RBC)

DGP	n	AI+adjustment 1%	5%	10%	IK+adjustment 1%	5%	10%	CCT+adjustment 1%	5%	10%	DPI 1%	5%	10%
Power1	1000	0.088	0.225	0.312	0.048	0.143	0.222	0.031	0.09	0.15	0.038	0.102	0.162
	2000	0.228	0.439	0.551	0.171	0.316	0.423	0.081	0.205	0.292	0.071	0.174	0.263
	4000	0.562	0.749	0.816	0.425	0.629	0.719	0.281	0.486	0.609	0.174	0.354	0.487
	8000	0.918	0.964	0.977	0.83	0.935	0.957	0.666	0.831	0.877	0.485	0.658	0.743
Power2	1000	0.045	0.133	0.214	0.008	0.046	0.096	0.006	0.045	0.087	0.015	0.054	0.094
	2000	0.108	0.266	0.384	0.052	0.156	0.244	0.023	0.097	0.178	0.037	0.097	0.179
	4000	0.316	0.56	0.672	0.19	0.399	0.519	0.099	0.248	0.376	0.083	0.215	0.316
	8000	0.744	0.889	0.927	0.587	0.793	0.857	0.368	0.598	0.707	0.225	0.427	0.550
Power3	1000	0.065	0.159	0.273	0.026	0.107	0.171	0.014	0.061	0.118	0.022	0.091	0.142
	2000	0.129	0.306	0.439	0.091	0.209	0.319	0.048	0.128	0.221	0.051	0.151	0.242
	4000	0.368	0.581	0.693	0.205	0.421	0.548	0.143	0.321	0.426	0.122	0.277	0.383
	8000	0.751	0.888	0.933	0.597	0.781	0.859	0.432	0.64	0.734	0.264	0.490	0.606
Power4	1000	0.025	0.101	0.171	0.005	0.036	0.078	0.004	0.027	0.059	0.014	0.055	0.095
	2000	0.055	0.175	0.263	0.027	0.092	0.188	0.016	0.074	0.143	0.034	0.080	0.148
	4000	0.113	0.265	0.403	0.073	0.201	0.324	0.03	0.138	0.23	0.036	0.131	0.216
	8000	0.324	0.545	0.681	0.233	0.438	0.569	0.121	0.283	0.425	0.084	0.245	0.349

Appendix G. Additional Empirical Results for Section 5

Table G.1. Jump Size of the Propensity Score (different choice of h)

	3	5	AL	IK	CCT
Grade 4					
Cut-off 40	0.60	0.65	0.80	0.60	0.77
Cut-off 80	0.47	0.48	0.64	0.51	0.56
Cut-off 120	0.30	0.44	0.56	0.37	0.55
Grade 5					
Cut-off 40	0.53	0.47	0.66	0.47	0.47
Cut-off 80	0.42	0.42	0.55	0.42	0.49
Cut-off 120	0.31	0.29	0.44	0.31	0.42

Table G.2. Testing Results for Israeli School Data: p-values, $\xi = 0.0316$

	3	5	AL	IK	CCT	MSE-RBC	CER-RBC
g4math							
Cut-off 40	0.986	0.934	0.764	0.978	0.968	0.975	0.974
Cut-off 80	0.909	0.865	0.715	0.944	0.888	0.776	0.973
Cut-off 120	0.443	0.702	0.665	0.604	0.568	0.610	0.646
g4verb							
Cut-off 40	0.928	0.627	0.465	0.641	0.529	0.574	0.455
Cut-off 80	0.911	0.883	0.185	0.906	0.720	0.300	0.855
Cut-off 120	0.935	0.683	0.474	0.730	0.186	0.222	0.131
g5math							
Cut-off 40	0.876	0.282	0.482	0.631	0.609	0.903	0.241
Cut-off 80	0.516	0.446	0.930	0.482	0.765	0.814	0.708
Cut-off 120	0.939	0.827	0.626	0.883	0.838	0.832	0.731
g5verb							
Cut-off 40	0.594	0.893	0.953	0.900	0.938	0.960	0.957
Cut-off 80	0.510	0.692	0.504	0.519	0.929	0.956	0.979
Cut-off 120	0.696	0.811	0.601	0.699	0.774	0.729	0.762

TABLE G.3. Testing Results for Israeli School Data: p-values, $\xi = 0.1706$

	3	5	AL	IK	CCT	MSE-RBC	CER-RBC
g4math							
Cut-off 40	0.986	0.934	0.945	0.978	0.959	0.961	0.965
Cut-off 80	0.909	0.865	0.713	0.944	0.878	0.763	0.953
Cut-off 120	0.443	0.702	0.660	0.565	0.540	0.599	0.646
g4verb							
Cut-off 40	0.924	0.627	0.451	0.637	0.517	0.571	0.469
Cut-off 80	0.911	0.883	0.185	0.906	0.688	0.281	0.836
Cut-off 120	0.935	0.683	0.471	0.730	0.183	0.249	0.125
g5math							
Cut-off 40	0.861	0.275	0.481	0.623	0.600	0.887	0.236
Cut-off 80	0.516	0.429	0.916	0.479	0.762	0.797	0.714
Cut-off 120	0.939	0.827	0.624	0.883	0.836	0.808	0.746
g5verb							
Cut-off 40	0.594	0.893	0.953	0.934	0.938	0.944	0.946
Cut-off 80	0.510	0.671	0.496	0.513	0.946	0.946	0.974
Cut-off 120	0.696	0.811	0.594	0.699	0.757	0.740	0.852

TABLE G.4. Testing Results for Israeli School Data: p-values, $\xi = 0.5$

	3	5	AL	IK	CCT	MSE-RBC	CER-RBC
g4math							
Cut-off 40	0.984	0.934	0.940	0.978	0.950	0.957	0.956
Cut-off 80	0.907	0.853	0.832	0.936	0.893	0.774	0.956
Cut-off 120	0.443	0.683	0.633	0.557	0.519	0.592	0.580
g4verb							
Cut-off 40	0.907	0.599	0.450	0.637	0.499	0.503	0.422
Cut-off 80	0.907	0.880	0.165	0.906	0.760	0.266	0.939
Cut-off 120	0.935	0.668	0.449	0.719	0.164	0.194	0.130
g5math							
Cut-off 40	0.854	0.678	0.461	0.788	0.829	0.917	0.832
Cut-off 80	0.499	0.419	0.913	0.466	0.749	0.785	0.691
Cut-off 120	0.931	0.812	0.591	0.873	0.818	0.763	.732
g5verb							
Cut-off 40	0.955	0.875	0.946	0.926	0.936	0.945	0.945
Cut-off 80	0.499	0.664	0.930	0.504	0.938	0.946	0.974
Cut-off 120	0.665	0.795	0.708	0.688	0.750	0.673	0.825

TABLE G.5. Testing Results for Colombia's SR Data: p-values ($\xi = 0.00999$, full table)

Outcome variables	MPV Bandwidths 2	3	4	Other Bandwidth Choices AI	IK	CCT	MSE RBC	CER RBC
Risk protection, consumption smoothing and portfolio choice								
Individual inpatient medical spending	0.53	0.79	0.85	0.56	0.95	0.96	0.63	0.86
Individual outpatient medical spending	0.95	0.87	0.85	0.01	0.61	0.89	0.92	0.87
Variability of individual inpatient medical spending	0.50	0.79	0.87	0.66	0.94	0.96	0.67	0.86
Variability of individual outpatient medical spending	0.91	0.95	0.99	0.83	0.68	0.97	0.98	0.94
Individual education spending	0.15	0.19	0.17	0.02	0.90	0.09	0.21	0.14
Household education spending	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00
Total spending on food	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00
Total monthly expenditure	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Has car	0.97	0.76	0.87	0.99	0.72	0.99	0.87	0.94
Has radio	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Medical care use								
Preventive physician visit	0.62	0.99	1.00	1.00	0.37	0.98	0.50	0.75
Number of growth development checks last year	0.73	0.93	0.96	0.92	0.65	0.99	0.85	0.96
Curative care use	0.98	0.96	0.96	0.98	0.97	0.95	0.99	0.94
Primary care	0.92	0.92	0.94	0.99	0.96	0.95	0.96	0.88
Medical visit-specialist	0.98	0.93	0.73	0.93	0.89	0.63	0.83	0.81
Hospitalization	0.99	1.00	1.00	1.00	0.98	1.00	1.00	1.00
Medical visit for chronic disease	0.15	0.49	0.72	0.53	0.09	0.64	0.64	0.73
Curative care use among children	0.95	0.98	0.98	1.00	0.95	0.99	0.99	0.99
Health status								
Child days lost to illness	0.60	0.67	0.80	0.76	0.66	0.86	0.76	0.83
Cough, fever, diarrhea	0.99	1.00	1.00	1.00	0.99	1.00	1.00	1.00
Any health problem	0.99	1.00	0.98	0.99	0.99	0.99	0.99	0.99
Birthweight (KG)	0.90	100	1.00	0.99	0.90	1.00	1.00	1.00
Behavioral distortions								
Drank alcohol during pregnancy	0.42	0.74	0.87	0.93	0.19	0.93	0.91	0.94
Number of drinks per week during pregnancy	0.78	0.88	0.91	0.85	0.75	0.84	0.85	0.87
Months child breastfed	0.94	0.95	0.92	0.86	0.94	0.87	0.95	0.90
Folic acid during pregnancy	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Number months folic acid during pregnancy	0.92	0.95	0.92	0.96	0.94	0.75	0.85	0.80
Contributory regime enrollment (ECV)	0.55	0.54	0.37	0.00	0.74	0.32	0.49	0.61
Contributory regime enrollment (DHS)	0.97	0.99	0.99	1.00	0.63	1.00	1.00	1.00
Other insurance (ECV)	0.89	0.93	0.92	0.87	0.81	0.90	0.94	0.92
Other insurance (DHS)	0.91	0.96	0.97	0.95	0.88	0.96	0.96	0.97
Uninsured (ECV)	0.67	0.68	0.45	0.07	0.75	0.68	0.69	0.68
Uninsured (DHS)	0.99	1.00	1.00	1.00	0.79	0.97	0.83	0.91

TABLE G.6. Testing Results for Colombia's SR Data: p-values ($\xi = 0.0316$, full table)

Outcome variables	MPV Bandwidths 2	3	4	Other Bandwidth Choices AI	IK	CCT	MSE RBC	CER RBC
Risk protection, consumption smoothing and portfolio choice								
Individual inpatient medical spending	0.53	0.78	0.85	0.56	0.94	0.95	0.60	0.87
Individual outpatient medical spending	0.93	0.85	0.82	0.01	0.56	0.84	0.87	0.87
Variability of individual inpatient medical spending	0.49	0.77	0.86	0.64	0.93	0.95	0.64	0.85
Variability of individual outpatient medical spending	0.87	0.93	0.97	0.95	0.62	0.95	0.92	0.91
Individual education spending	0.15	0.18	0.17	0.02	0.89	0.09	0.18	0.14
Household education spending	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00
Total spending on food	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00
Total monthly expenditure	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Has car	0.97	0.76	0.86	0.99	0.72	0.99	0.88	0.95
Has radio	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Medical care use								
Preventive physician visit	0.62	0.99	1.00	1.00	0.37	0.98	0.52	0.73
Number of growth development checks last year	0.72	0.93	0.96	0.92	0.64	0.99	0.88	0.96
Curative care use	0.98	0.96	0.96	0.98	0.97	0.95	0.97	0.95
Primary care	0.92	0.92	0.94	0.99	0.96	0.95	0.94	0.88
Medical visit-specialist	0.98	0.93	0.73	0.93	0.89	0.63	0.79	0.80
Hospitalization	0.99	1.00	1.00	1.00	0.98	1.00	1.00	1.00
Medical visit for chronic disease	0.15	0.49	0.72	0.53	0.09	0.64	0.63	0.72
Curative care use among children	0.96	0.98	0.97	1.00	0.95	1.00	0.99	0.98
Health status								
Child days lost to illness	0.60	0.67	0.80	0.76	0.66	0.86	0.77	0.82
Cough, fever, diarrhea	0.99	1.00	1.00	1.00	0.99	1.00	1.00	1.00
Any health problem	0.99	1.00	0.98	0.99	0.99	0.99	0.99	0.99
Birthweight (KG)	0.90	100	1.00	0.99	0.90	1.00	1.00	1.00
Behavioral distortions								
Drank alcohol during pregnancy	0.42	0.74	0.87	0.93	0.19	0.93	0.89	0.94
Number of drinks per week during pregnancy	0.78	0.88	0.90	0.85	0.72	0.84	0.81	0.86
Months child breastfed	0.94	0.95	0.92	0.86	0.94	0.87	0.95	0.90
Folic acid during pregnancy	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Number months folic acid during pregnancy	0.92	0.95	0.92	0.96	0.94	0.75	0.87	0.79
Contributory regime enrollment (ECV)	0.55	0.54	0.37	0.00	0.74	0.32	0.50	0.65
Contributory regime enrollment (DHS)	0.97	0.99	0.99	1.00	0.63	1.00	1.00	1.00
Other insurance (ECV)	0.89	0.93	0.92	0.87	0.81	0.90	0.94	0.93
Other insurance (DHS)	0.91	0.96	0.97	0.95	0.88	0.96	0.95	0.98
Uninsured (ECV)	0.67	0.68	0.45	0.07	0.75	0.68	0.68	0.69
Uninsured (DHS)	0.99	1.00	1.00	1.00	0.79	0.97	0.83	0.91

TABLE G.7. Testing Results for Colombia's SR Data: p-values ($\xi = 0.1706$, full table)

	MPV Bandwidths			Other Bandwidth Choices				
Outcome variables	2	3	4	AI	IK	CCT	MSE RBC	CER RBC
Risk protection, consumption smoothing and portfolio choice								
Individual inpatient medical spending	0.43	0.68	0.76	0.99	0.88	0.90	0.5	0.82
Individual outpatient medical spending	0.85	0.84	0.78	0.64	0.41	0.80	0.84	0.83
Variability of individual inpatient medical spending	0.37	0.64	0.79	0.97	0.83	0.91	0.61	0.85
Variability of individual outpatient medical spending	0.77	0.89	0.95	0.93	0.35	0.91	0.90	0.90
Individual education spending	0.14	0.16	0.16	0.08	0.87	0.26	0.17	0.14
Household education spending	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00
Total spending on food	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00
Total monthly expenditure	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Has car	0.97	0.76	0.86	0.99	0.72	0.99	0.87	0.95
Has radio	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Medical care use								
Preventive physician visit	0.62	0.99	1.00	1.00	0.37	0.98	0.50	0.75
Number of growth development checks last year	0.77	0.89	0.95	0.99	0.82	0.98	0.90	.95
Curative care use	0.98	0.96	0.96	0.98	0.97	0.95	0.99	0.95
Primary care	0.92	0.92	0.94	0.99	0.96	0.95	0.95	0.88
Medical visit-specialist	0.98	0.93	0.73	0.93	0.89	0.63	0.79	0.78
Hospitalization	0.99	1.00	1.00	1.00	0.98	1.00	1.00	1.00
Medical visit for chronic disease	0.15	0.49	0.72	0.53	0.09	0.64	0.65	0.73
Curative care use among children	0.95	0.98	0.98	1.00	0.95	0.99	0.99	0.99
Health status								
Child days lost to illness	0.60	0.67	0.80	0.76	0.66	0.86	0.73	0.83
Cough, fever, diarrhea	0.99	1.00	1.00	1.00	0.99	1.00	1.00	1.00
Any health problem	0.99	1.00	0.98	0.99	0.99	0.99	0.99	0.99
Birthweight (KG)	0.90	100	1.00	0.99	0.90	1.00	1.00	1.00
Behavioral distortions								
Drank alcohol during pregnancy	0.42	0.74	0.87	0.93	0.19	0.93	0.88	0.92
Number of drinks per week during pregnancy	0.73	0.85	0.88	0.83	0.69	0.80	0.76	0.80
Months child breastfed	0.94	0.95	0.92	0.86	0.94	0.87	0.96	0.90
Folic acid during pregnancy	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Number months folic acid during pregnancy	0.92	0.95	0.92	0.96	0.94	0.75	0.86	0.80
Contributory regime enrollment (ECV)	0.55	0.54	0.37	0.00	0.74	0.32	0.51	0.57
Contributory regime enrollment (DHS)	0.97	0.99	0.99	1.00	0.63	1.00	1.00	1.00
Other insurance (ECV)	0.89	0.93	0.92	0.87	0.81	0.90	0.93	0.93
Other insurance (DHS)	0.91	0.96	0.97	0.95	0.88	0.96	0.96	0.98
Uninsured (ECV)	0.67	0.68	0.45	0.07	0.75	0.68	0.69	0.68
Uninsured (DHS)	0.99	1.00	1.00	1.00	0.79	0.97	0.84	0.92

TABLE G.8. Testing Results for Colombia's SR Data: p-values ($\xi = 0.5$, full table)

Outcome variables	MPV Bandwidths 2	3	4	Other Bandwidth Choices AI	IK	CCT	MSE RBC	CER RBC
Risk protection, consumption smoothing and portfolio choice								
Individual inpatient medical spending	0.52	0.68	0.71	0.89	0.76	0.71	0.58	0.76
Individual outpatient medical spending	0.72	0.84	0.93	0.22	0.23	0.90	0.71	0.90
Variability of individual inpatient medical spending	0.37	0.61	0.69	0.89	0.63	0.76	0.60	0.76
Variability of individual outpatient medical spending	0.40	0.75	0.76	0.54	0.18	0.78	0.66	0.74
Individual education spending	0.10	0.11	0.11	0.29	0.78	0.19	0.13	0.10
Household education spending	0.00	0.00	0.00	0.03	0.02	0.00	0.00	0.00
Total spending on food	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00
Total monthly expenditure	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Has car	0.97	0.76	0.86	0.99	0.72	0.99	0.86	0.95
Has radio	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Medical care use								
Preventive physician visit	0.62	0.99	1.00	1.00	0.37	0.98	0.51	0.76
Number of growth development checks last year	0.81	0.89	0.97	0.98	0.88	0.99	0.90	0.96
Curative care use	0.98	0.96	0.96	0.98	0.97	0.95	0.98	0.94
Primary care	0.92	0.92	0.94	0.99	0.96	0.95	0.96	0.90
Medical visit-specialist	0.96	0.92	0.72	0.91	0.87	0.63	0.80	0.79
Hospitalization	0.99	1.00	1.00	1.00	0.98	1.00	1.00	1.00
Medical visit for chronic disease	0.15	0.49	0.72	0.53	0.09	0.64	0.63	0.74
Curative care use among children	0.96	0.99	0.98	1.00	0.94	0.99	0.99	0.99
Health status								
Child days lost to illness	0.60	0.67	0.80	0.76	0.66	0.86	0.75	0.82
Cough, fever, diarrhea	0.99	1.00	1.00	1.00	0.99	1.00	1.00	1.00
Any health problem	0.99	1.00	0.98	0.99	0.99	0.99	0.99	0.99
Birthweight (KG)	0.90	100	1.00	0.99	0.90	1.00	0.91	0.94
Behavioral distortions								
Drank alcohol during pregnancy	0.42	0.74	0.87	0.93	0.19	0.93	0.89	0.94
Number of drinks per week during pregnancy	0.66	0.80	0.85	0.73	0.65	0.75	0.72	0.78
Months child breastfed	0.94	0.95	0.91	0.86	0.94	0.87	0.95	0.92
Folic acid during pregnancy	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Number months folic acid during pregnancy	0.92	0.95	0.92	0.96	0.94	0.75	0.88	0.80
Contributory regime enrollment (ECV)	0.55	0.54	0.37	0.00	0.74	0.32	0.50	0.63
Contributory regime enrollment (DHS)	0.97	0.99	0.99	1.00	0.63	1.00	1.00	1.00
Other insurance (ECV)	0.86	0.91	0.91	0.86	0.80	0.89	0.94	0.92
Other insurance (DHS)	0.90	0.96	0.97	0.95	0.87	0.96	0.96	0.97
Uninsured (ECV)	0.67	0.68	0.45	0.07	0.75	0.68	0.69	0.70
Uninsured (DHS)	0.99	1.00	1.00	1.00	0.79	0.97	0.83	0.90

TABLE G.9. Testing Results for Colombia's SR Data by Regions ($\xi = 0.00999$)

	MPV bandwidths			Other bandwidth choice		
	2	3	4	AI	IK	CCT
Atlantica						
Household education spending	0.001	0.001	0.001	0.000	0.000	0.001
Total spending on food	0.009	0.008	0.026	0.000	0.015	0.020
Total monthly expenditure	0.000	0.001	0.000	0.000	0.000	0.000
Oriental						
Household education spending	0.000	0.000	0.000	0.000	0.000	0.002
Total spending on food	0.000	0.001	0.000	0.000	0.001	0.002
Total monthly expenditure	n.a.*	n.a.	n.a.	n.a.	n.a.	n.a.
Central						
Household education spending	0.000	0.098	0.058	0.000	0.000	0.000
Total spending on food	0.000	0.002	0.001	0.001	0.000	0.021
Total monthly expenditure	0.000	0.007	0.008	0.000	0.000	0.001
Pacifica						
Household education spending	0.001	0.147	0.073	0.000	0.043	0.003
Total spending on food	0.150	0.237	0.236	0.013	0.107	0.385
Total monthly expenditure	0.091	0.347	0.231	0.002	0.071	0.125
Bogota						
Household education spending	0.000	0.000	0.000	0.000	0.014	0.000
Total spending on food	0.000	0.000	0.001	0.003	0.002	0.000
Total monthly expenditure	0.000	0.000	0.000	0.000	0.000	0.000
Territorios Nacionales						
Household education spending	0.085	0.247	0.063	0.000	0.037	0.090
Total spending on food	0.029	0.310	0.032	0.000	0.057	0.281
Total monthly expenditure	0.227	0.271	0.349	0.001	0.364	0.752

$*$: not available due to small sample size.

TABLE G.10. Subsample Sizes by Regions

	Household Edu. Spending	Total Spending on Food	Total Monthly Exp.
Atlantica	3969	3969	1480
Oriental	1496	1496	452
Central	5341	5318	2728
Pacifica	6370	6370	3203
Bogota	43656	41108	14634
Territorios Nacionales	1137	1137	643

TABLE G.11. Sample Sizes and Bandwidths for the Israeli School Data

		3		5		AI		IK		CCT	
g4math											
Cut-off 40 ($n = 984$)	(n_-, n_+)	17	67	26	93	102	302	23	84	89	227
	(h_-, h_+)	3	3	5	5	11.1	15.0	3.8	3.9	10.6	10.4
Cut-off 80 ($n = 1376$)	(n_-, n_+)	29	45	76	71	292	142	29	45	206	107
	(h_-, h_+)	3	3	5	5	15.0	9.3	2.8	2.8	10.5	10.6
Cut-off 120 ($n = 976$)	(n_-, n_+)	27	20	66	34	189	66	47	34	117	60
	(h_-, h_+)	3	3	5	5	15.0	10.4	4.0	4.2	8.7	9.0
g4verb											
Cut-off 40 ($n = 984$)	(n_-, n_+)	17	67	26	93	57	302	23	84	89	227
	(h_-, h_+)	3	3	5	5	7.7	15.0	4.0	4.0	11.0	10.8
Cut-off 80 ($n = 1376$)	(n_-, n_+)	29	45	76	71	270	142	55	54	206	107
	(h_-, h_+)	3	3	5	5	13.7	9.7	3.2	3.2	10.2	10.4
Cut-off 120 ($n = 976$)	(n_-, n_+)	27	20	66	34	189	93	66	34	138	66
	(h_-, h_+)	3	3	5	5	15.0	13.3	4.3	4.4	10.3	10.7
g5math											
Cut-off 40 ($n = 983$)	(n_-, n_+)	19	77	38	112	143	328	29	94	47	130
	(h_-, h_+)	3	3	5	5	15.0	15.0	4.0	4.0	5.6	5.5
Cut-off 80 ($n = 1359$)	(n_-, n_+)	59	44	80	86	285	223	72	65	201	150
	(h_-, h_+)	3	3	5	5	15.0	15.0	3.9	4.0	10.4	10.6
Cut-off 120 ($n = 905$)	(n_-, n_+)	36	22	61	31	166	56	49	25	109	56
	(h_-, h_+)	3	3	5	5	15.0	8.1	3.7	3.9	8.1	8.4
g5verb											
Cut-off 40 ($n = 983$)	(n_-, n_+)	19	77	38	112	58	268	38	112	70	184
	(h_-, h_+)	3	3	5	5	6.4	11.5	4.2	4.1	7.2	7.0
Cut-off 80 ($n = 1359$)	(n_-, n_+)	59	44	80	86	285	223	72	65	201	154
	(h_-, h_+)	3	3	5	5	15.0	15.0	3.7	3.8	10.5	10.7
Cut-off 120 ($n = 905$)	(n_-, n_+)	36	22	61	31	166	45	49	25	79	45
	(h_-, h_+)	3	3	5	5	15.0	6.8	3.2	3.3	6.7	7.0

Note: h_- and h_+ denote the specified bandwidths to the left and right of the cut-off, respectively. The data driven bandwidths presented in this table (AI, IK, and CCT) are undersmoothed by multiplying $(\sum_{i=1}^{n} 1\{R_i < r_0\})^{1/5-1/4.5}$ and $(\sum_{i=1}^{n} 1\{R_i \geq r_0\})^{1/5-1/4.5}$, respectively. We set the upper-bound of the data driven bandwidths at 15. n_- and n_+ denote the number of observations with values of the running variable in $(r_0 - h_-, r_0)$ and $[r_0, r_0 + h_+)$, respectively.

TABLE G.12. Sample Sizes and Bandwidths for Colombia's SR Data

Outcomes		2		3		4		AI		IK		CCT	
HES	(n_-, n_+)	1701	2521	2474	3783	3034	5204	3484	18798	1082	1423	3586	6611
	(h_-, h_+)	2	2	3	3	4	4	54.99	11.8	1.11	1.05	5.23	4.96
TSF	(n_-, n_+)	1664	2432	2420	3655	2979	5034	3410	18247	1050	1384	3512	6385
	(h_-, h_+)	2	2	3	3	4	4	3.78	23.5	1.36	1.29	3.70	3.51
TME	(n_-, n_+)	402	564	567	828	643	1136	732	4867	285	314	754	1398
	(h_-, h_+)	2	2	3	3	4	4	6.08	8.32	0.99	0.92	2.12	1.98

HES: Household Education Spending; TSF: Total Spending on Food; TME: Total Monthly Expenditure.
Note: h_- and h_+ denote the specified bandwidths to the left and right of the cut-off, respectively. The data driven bandwidths presented in this table (AI, IK, and CCT) are undersmoothed by multiplying $(\sum_{i=1}^{n} 1\{R_i \leq r_0\})^{1/5-1/4.5}$ and $(\sum_{i=1}^{n} 1\{R_i > r_0\})^{1/5-1/4.5}$, respectively. We set the upper-bound of the data driven bandwidths at 15. n_- and n_+ denote the number of observations with values of the running variable in $(r_0 - h_-, r_0)$ and $[r_0, r_0 + h_+)$, respectively.

FIGURE G.1. Estimated compliers' outcome density: Household education spending

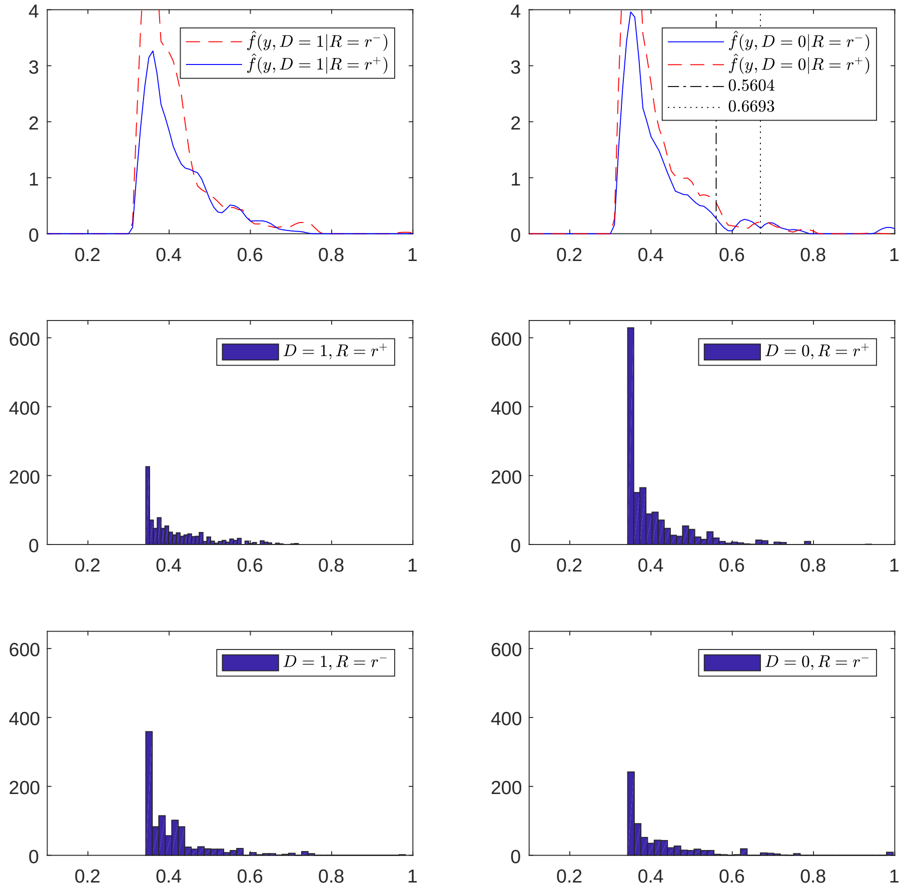

TABLE G.13. Observations in the maximizing interval ($h^+ = h^- = 2$): Household edu. spending

Household education spending	# of observations		
Subsample of	All	$\cap\{0.5604 \leq Y \leq 0.6693\}$	Ratio
$\{0 \leq R < h^+\} \cap \{D = 0\} \Leftrightarrow \mathbf{N} \cup \mathbf{C}$	1563	43	2.75%
$\{h^- < R < 0\} \cap \{D = 0\} \Leftrightarrow \mathbf{N}$	690	25	3.62%

FIGURE G.2. Estimated compliers' outcome density: Total monthly spending

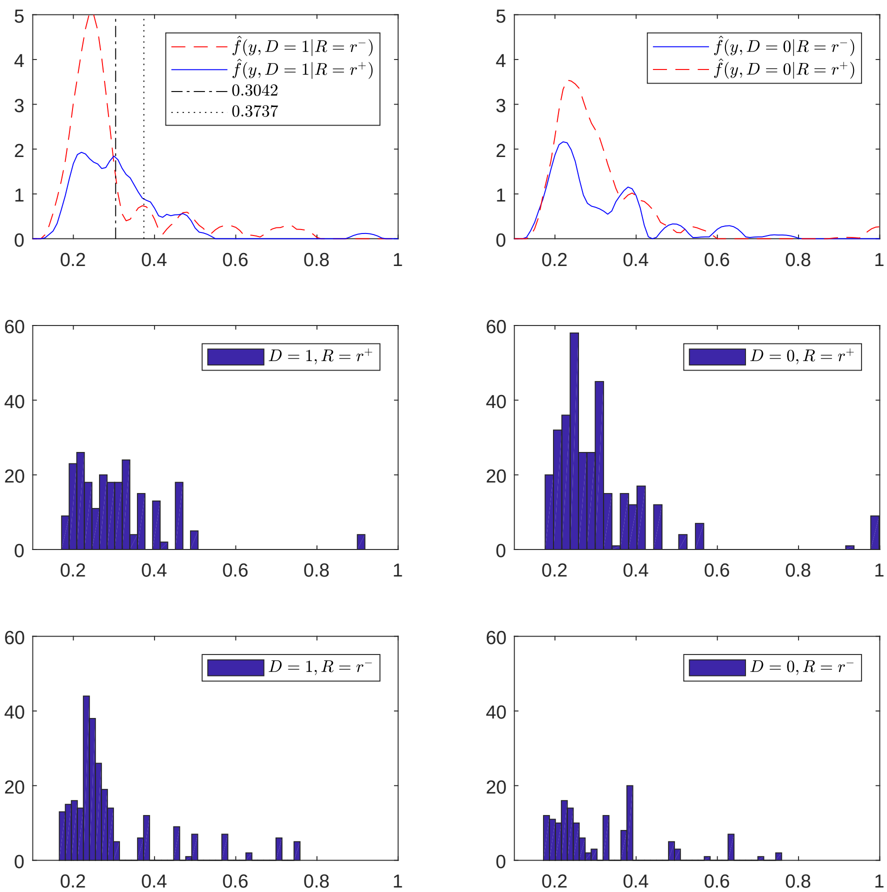

TABLE G.14. Observations in the maximizing interval ($h^+ = h^- = 2$):Total monthly expenditure

Total monthly expenditure	# of observations		
Subsample of	All	$\cap\{0.3042 \leq Y \leq 0.3737\}$	Ratio
$\{0 \leq R < h^+\} \cap \{D=1\} \Leftrightarrow \mathbf{A}$	228	61	26.7%
$\{h^- < R < 0\} \cap \{D=1\} \Leftrightarrow \mathbf{A} \cup \mathbf{C}$	259	6	2.32%

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